Optimal strategies from forward versus classical utilities

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A thesis submitted in partial fulfillment of the MSc in

Mathematical and Computational Finance

Trinity 2019

Acknowledgements

I want to thank my supervisor, who dedicated a lot of his precious time to discuss with me various issues that arised and contributed with his expertise to give me a clear guideline and assistance to overcome any obstacles I encountered throughout the making of this dissertation.

Further, I would like to thank my friends from the course and in my college for giving me some helpful new perspectives and ideas when discussing our work, for pushing through those long nights in the library together and for providing some welcome distraction throughout this term.

Lastly, I want to express my deepest gratitude to my family for always being there for me, for the permanent encouragement and for the unconditional support. Without you none of this would have been possible.

Abstract

Forward utility theory has been established independently by Musiela & Zariphopoulou [43, 44] and Henderson & Hobson [18] as an alternative to expected utility theory in the assessment of optimal investment strategies. After its introduction, vast amounts of literature have analysed and extended the theory in various directions. This thesis aims to give an accurate overview of the current theory of forward utilities, summarise, unify and generalise the main results from the literature, and assess its implications for the optimal strategies. We introduce a new and very general class of forward utilities called *Itô-type forward performance processes*, which contains all forward utility functions that allow for an Itô decomposition, and thus all of the most widely used forward utilities from the literature. This definition allows us to generalise the results for the optimal investment strategies under such forward performance criteria, and we find that these optimal strategies are always myopic if the Itô volatility process is not wealth-dependent. Furthermore, we find that these optimal strategies ignore unhedgeable risk factors, except when the external stochastic factor is introduced into the volatility process of the forward utility. We then extend the definition of Itô-type forward utilities to investment and consumption problems in a consistent way, which provides a general framework for future extensions of the theory in this direction. We give an explicit example of an Itô-type forward utility pair of investment and consumption and compare its optimal strategies to their classical counterparts. Moreover, we establish that the indifference pricing formula of a European-type random endowment for (time-monotone) forward exponential utilities has the same structural form as in the classical framework, but the minimal entropy measure appearing in the classical approach is replaced by the minimal martingale measure in the forward approach. Lastly, we point out some potential areas for future research that arise from our analysis, which can build on the generalisation and unification of the forward utility theory provided herein.

Contents

1 Introduction

In this thesis, we examine the theory of *forward performance processes*[1], which has been introduced independently by Musiela & Zariphopoulou [43, 44] and Henderson & Hobson [18] and provides a new framework to assess optimal investor behaviour for some future investment period. Classically, this issue has been examined by the concept of expected utility theory and stochastic control techniques (see, e.g., the books by Pham [50] and Rogers [53] for an overview of the standard concepts). Subsequently, a vast amount of literature has emerged which studied forward performance processes and extended the theory in various directions. The objective of this thesis is to provide a comprehensive overview and summarise, unify and generalise the main results from the literature to provide an accurate understanding of the existing theory of forward utility processes. We further aim to compare the resulting optimal portfolios from the forward utility theory to the ones known from the classical theory, first on a very general level in an incomplete Itô-process market, and then consider different examples of popular market models to get explicit solutions. We will analyse the optimal portfolios from a financial perspective and in particular discuss the implications for risk management. We also consider an optimal investment problem with consumption, which has rarely been studied so far in the forward utility framework, and, moreover, look at the theory of indifference pricing with forward utilities, which is an important concept for situations when the investor is faced with a random future endowment that cannot be perfectly hedged, so that the investor must decide how to incorporate this into her investment policy. Lastly, we want to look at open questions that arise from our study to highlight interesting areas for potential future research.

Our contribution is threefold: Firstly, we summarise and unify the key findings of the literature to give an accurate overview of the current theory of forward utility functions. By introducing a new class of so-called *Itô-type forward utility functions* (Definition 2) we are able to generalise results regarding the optimal investment strategies (Theorem 4.2) and can draw the insightful conclusions that - for this general class - external, unhedgeable risk factors are ignored in the optimal strategies (Corollary 4.2.2), with the only exception when the external stochastic factor is introduced into the volatility process, and in the case that the volatility process of the Itô-type forward utility is not wealth-dependent, the optimal portfolio is always the *myopic portfolio* (Corollary 4.2.1). The generality of these results is highlighted by the fact that our newly defined class contains all forward utility functions that allow an Itô decomposition, which is assured by some of the main characterisation results from the literature, and thus all classes of forward performance performance processes that allow for an explicit representation which have appeared in the literature thus far. Secondly, we look at the natural extension to optimal investment problems by allowing for consumption and summarise the (scarce) literature on forward utilities of investment and consumption. We again manage to use characterisation results from the literature to generalise our definition of *Itô-type forward performance processes* to the case with investment and consumption (Definition 4) in a way such that our previous definition is contained as the special case when the utility from consumption is set equal to zero. This allows us to reformulate the result established in El Karoui et al. [13] and Källblad [25] regarding the optimal investment and consumption policies in terms

[1]Subsequently also called forward utility functions, or forward utilities.

of this newly defined general class (Theorem 5.2). We construct an explicit, power-type member of this class (Lemma 3), compute the resulting optimal investment and consumption policy in a Black-Scholes market model and compare it with the optimal finite and infinite horizon *Merton strategies*. We find that a scaling parameter in our forward utility process allows us to obtain any fraction of wealth as the optimal consumption path and that we can thus replicate the infinite horizon Merton strategy but, due to the lack of horizon dependence in the forward approach, not the finite horizon strategy. However, we also highlight that a general class of forward utility pairs of investment and consumption, which allows for an explicit representation and tractable solutions, is yet to be characterised and remains an open problem for future research. Thirdly, we show that using so-called *time-monotone* exponential forward performance criteria in a general incomplete Itô-process market, the structure of the classical formula for the indifference price of a European claim is recovered, but the minimal entropy measure appearing in the classical formula is replaced by the minimal martingale measure in the new forward formula (Theorem 6.2), again indicating that unhedgeable risk factors are ignored. As a consequence, we can deduce that the marginal utility-based price of such a claim is the expectation of the payoff with respect to the minimal martingale measure (Corollary 6.2.1), as opposed to the expected payoff under the minimal entropy measure in the classical framework. Results of this nature have appeared in previous works, but not in the definitive and general form we establish herein. Lastly, we emphasise open questions and possible areas for future research, such as duality theory, which is a huge and important field in the classical theory, but little is known about the dual side of forward utilities.

The remainder of this thesis is structured as follows. In Section 2 we set up the general Itô-process market model which will be used throughout this thesis. Section 3 introduces the concept of forward utility functions, gives a comprehensive overview of the existing literature and highlights the main achievements that have been made thus far. In Section 4, we consider the most widely studied application of forward utility functions, namely optimal investment problems, and generalise existing results with respect to optimal investment policies from the literature by means of our newly defined class of Itô-type forward utilities and consider specific examples of market models to compare the optimal strategies of the classical versus forward approach. We also highlight its financial implications, in particular for risk management. In Section 5 we consider the forward utility framework of optimal investment with consumption. After reviewing the existing literature which has covered this problem so far, we generalise the main results and give an explicit example so that we can again compare the optimal strategies of classical versus forward utilities in a basic Black-Scholes model. Section 6 examines the theory of utility indifference pricing of contingent claims and establishes a result that highlights the similarities and differences in the general pricing formulae for a random (European-type) claim in an incomplete Itô-process market for classical and (time-monotone) forward exponential utilities. Section 7 concludes by summarising the main results and discussing open questions and potential areas of future research by considering important concepts in the classical theory (such as duality theory or investment with consumption problems) for which little is known so far for the forward framework, and by pointing out some of the important newly arising ingredients in the forward performance framework, for which a deeper and more thorough understanding is needed.

2 Market model

Throughout this thesis we consider a financial market with d risky assets and one risk-free asset (which could be thought of as a bank account or a government bond of a quasi default-free economy). We model this stock market by considering the asset prices as being Itô processes on an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathcal{T}}$ for some time set $\mathcal{T}$, which may be finite ($\mathcal{T} = [0, T]$ for some finite time horizon T), or infinite ($\mathcal{T} = [0, \infty)$). The stock prices are driven by an m-dimensional Brownian motion, with $m \geq d$, which means that the price process of the i^{th} asset S^i evolves according to

$$dS_t^i = S_t^i(\mu_t^i dt + \sum_{j=1}^{m} \sigma_t^{ij} dW_t^j), \qquad i = 1, ..., d, \tag{1}$$

where $(\mu_t)_{t\in\mathcal{T}} \in \mathbb{R}^m$ and $(\sigma_t)_{t\in\mathcal{T}} \in \mathbb{R}_+^{d\times m}$ are $\mathbb{F}$-adapted processes and are such that $\int_0^t \|\mu_s\| + \|\sigma_s\|^2 ds < \infty$, almost surely, for all $t \in \mathcal{T}$. Equivalently, we can write this in matrix notation

$$dS_t = \text{diag}_d(S_t)(\mu_t dt + \sigma_t dW_t), \tag{2}$$

where $\text{diag}_d(x)$ denotes the $d \times d$ matrix whose diagonal is given by the vector $x \in \mathbb{R}^d$ and whose off-diagonal entries are zero. The riskless asset satisfies

$$dB_t = r_t B_t dt, \qquad B_0 = 1, \tag{3}$$

for the $\mathbb{F}$-adapted interest rate process $(r_t)_{t\in\mathcal{T}}$ with $\int_0^t |r_s| ds < \infty$, almost surely, $t \in \mathcal{T}$.

Standing assumptions We will make following standing assumptions and assume them to hold throughout this thesis:
Assumption 1: Without loss of generality (w.l.o.g.), we assume that $r_t = 0$. For the case of a non-zero interest rate the results in this thesis still hold with any price or wealth-process being replaced by its discounted counterpart.
Assumption 2: A standard assumption in a general incomplete Itô-market model is that σ_t has full row-rank for every $t \in \mathcal{T}$, see e.g. Remark 4.10 in the seminal book by Karatzas & Shreve [29], which shows that if the rank of σ_t is $\hat{d} < d$, then we can find $\hat{d}$ assets which can replicate the remaining $d - \hat{d}$ assets. We can thus define a $\hat{d}$-dimensional financial market with a volatility matrix $\hat{\sigma}_t$ of full row-rank for all $t \in \mathcal{T}$. Therefore, we can make this assumption w.l.o.g..
Assumption 3: The process $(\lambda_t)_{t\in\mathcal{T}}$ defined below (4) satisfies $\int_0^t \|\lambda_s\|^2 ds < \infty$, almost surely, for all $t \in \mathcal{T}$.[2]

By the assumption that the rows of the volatility matrix are linearly independent, the matrix $(\sigma_t\sigma_t^{tr})^{-1}$ is well defined, where σ_t^{tr} denotes the transpose of the matrix σ_t, and we can define the relative risk process by

$$\lambda_t := \sigma_t^{tr}(\sigma_t\sigma_t^{tr})^{-1}\mu_t, \qquad t \in \mathcal{T}. \tag{4}$$

[2]We note here that in most papers treating forward utilities λ is assumed to be bounded. This unnecessarily restricts the choice of market models though, since models like the Heston model, or a model with a Gaussian market price of risk are excluded from the analysis then.

In many papers λ is referred to as *market price of risk*[3]. We remark though that the market price of risk is only unique if the market is complete ($m = d$). In the case of an incomplete market, i.e. $m > d$, λ is but one of the infinitely many solutions to the market price of risk equations (5). Any solution, which is a process $(q_t)_{t\in\mathcal{T}} \in \mathbb{R}^m$ satisfying

$$\sigma_t q_t = \mu_t, \qquad t \in \mathcal{T}, \tag{5}$$

defines a deflator

$$Z := \mathcal{E}(-q \cdot W)_{\cdot} = e^{\int_0^{\cdot} -q_s dW_s - \frac{1}{2}\int_0^{\cdot} q_s^2 ds} \tag{6}$$

so that ZS is a local martingale. In the case when Z is a true martingale (a sufficient condition for this is given by the Novikov condition), Z is the density process of an equivalent martingale measure (EMM). In fact, λ gives the density process of the minimal martingale measure, denoted by $\mathbb{Q}^M$. All other EMMs $\mathbb{Q}$ correspond to integrands q given by

$$q = \lambda + \nu, \tag{7}$$

with processes $(\nu_t)_{t\in\mathcal{T}} \in \mathbb{R}^m$ satisfying

$$\sigma_t \nu_t = \mathbf{0}_d, \qquad t \in \mathcal{T}. \tag{8}$$

We denote the set of EMMs by $\mathcal{M}$ and we will write $Z^{\mathbb{Q}}$ for the density process corresponding to $\mathbb{Q} \in \mathcal{M}$.

3 The theory of forward performance processes

We consider the general situation of an investor who wants to invest her initial capital x and is faced with the decision how to optimally allocate the capital to the different assets. We denote by π_t^i the present value of the amount of her wealth in asset i and by θ_t^i the fraction of her wealth that is invested in the i^{th} asset at time t. Then the investor's goal is to find the optimal self-financing trading strategy $\pi^i = (\pi_t^i)_{t\in\mathcal{T}}$ for $i = 0, ..., d$, where π^0 denotes the amount invested in the risk-free asset. We assume that the investor can trade continuously. The corresponding (controlled) wealth process is denoted by X^π with initial wealth $X_0^\pi = x$. We observe that $X_t^\pi = \sum_{i=0}^d \pi_t^i$, which has the dynamics

$$dX_t^\pi = \pi_t \sigma_t (\lambda_t dt + dW_t). \tag{9}$$

The classical approach to tackle the optimal investment problem has been introduced and solved by Merton [33, 34] and is given by first fixing an investment horizon T and a utility function[4] U at the end of the investment period. Throughout this thesis we assume a utility function to be a twice differentiable, increasing and strictly concave function satisfying the Inada conditions $\lim_{x\to\underline{x}} U'(x) = \infty, \lim_{x\to\infty} U'(x) = 0$, where $\underline{x}$

[3]A more accurate term, in fact, would be 'stock's market price of risk', as opposed to the market price of risk associated with other risk factors.

[4]For a concise overview of the general idea and the most important examples of utility functions see Gerber and Pafumi [16]; for a more comprehensive treatment we refer the reader to the book by Arrow [2].

denotes the lower boundary of the domain of U. The concavity reflects the risk-aversion of the investor. The agent's objective is to maximise expected terminal utility

$$\mathbb{E}[U(X_T^\pi)] \to \max!, \tag{10}$$

where the maximum is taken over the set of admissible trading strategies, which is denoted by $\mathcal{A}$ and contains all strategies $(\pi_t)_{0\le t\le T}$ which are adapted, self-financing, with wealth bounded from below and satisfy $\int_0^T \|\sigma_t\pi_t\|^2 dt < \infty$, almost surely. To solve problem (10) one typically defines the performance process

$$H^\pi(X_t^\pi, t) = \mathbb{E}[U(X_T^\pi)|\mathcal{F}_t], \qquad 0 \le t \le T,\ \pi \in \mathcal{A}. \tag{11}$$

Standard results from optimal control theory show that the performance process (11) is a supermartingale for any $\pi \in \mathcal{A}$ and a martingale for the optimal trading strategy π^* (see, for example, Section 3.3 in Pham [50] or Chapter 1 in Rogers [53]). The process $H(X_t^{\pi^*}, t) = \operatorname{ess\,sup}_{\pi\in\mathcal{A}} H^\pi(X_t^\pi, t)$ is called the value function process.

Musiela & Zariphopoulou in a series of papers ([43, 44, 45, 47]) have critiqued the notion that fixing at the outset the trading horizon as well as the investor's future preferences seems unnatural. They argue that future preferences can change over time as the market environment evolves and that it is rarely the case that an investment problem ends at a given point in time in the future, which moreover - from a pricing perspective in the theory of indifference pricing - restricts the set of claims that can be considered to the ones with a maturity smaller than T. For these reasons, they proposed a new framework to assess the performance of a trading strategy. They first introduce their so-called *forward performance process* in [43] for a binomial market model and extend it in [44] to the continuous case in an Itô-type market, where they define it as follows:

Definition 1. An $\mathbb{F}$-adapted process $\big(U_t(\cdot)\big)_{t\ge0}$ is a forward performance process, if

i) the mapping $x \mapsto U_t(x)$ is strictly concave and increasing, for all $t \ge 0$,

ii) for each $\pi \in \mathcal{A}$, $\mathbb{E}[|\ U_t(X_t^\pi)\ |] < \infty$, and

$$\mathbb{E}[U_s(X_s^\pi)|\mathcal{F}_t] \le U_t(X_t^\pi), \qquad s \ge t,$$

iii) there exists $\pi^* \in \mathcal{A}$, such that

$$\mathbb{E}[U_s(X_s^{\pi^*})|\mathcal{F}_t] = U_t(X_t^{\pi^*}), \qquad s \ge t,$$

iv) at $t = 0$, $U_0(x) = U(x)$ for some utility function U, $x \in \text{domain}(U)$.

In this approach, the investor specifies her utility function at the initial time, and the performance process is then constructed forward in time, allowing the investor to dynamically update her preferences as market information is revealed. Conditions ii) and iii) require that the forward performance process is a supermartingale for any admissible trading strategy and a martingale for the optimal one, thereby preserving the characteristics of the traditional value function. Also note that the forward performance process $U_t(\cdot)$ is defined for all $t \ge 0$ and is hence not restricted to a specific investment period. A similar framework has been introduced independently by Henderson & Hobson

[18] at the same time, who also require the supermartingale and martingale conditions for their so-called *horizon-unbiased utility functions*. The concept of forward utility functions is subsequently extended in a series of papers from various authors: In [44] Musiela & Zariphopoulou show how a certain class of forward utilities can be constructed by combining a differential input corresponding to the investor's risk-aversion with stochastic inputs which capture the changes in the market environment (see also Theorem 4.1 herein). In the special case when the stochastic inputs are non-random, the forward utility is non-random as well. This subclass of performance criteria has zero volatility and is called *time-monotone forward utilities*. It is further studied in [45] and [47]: In [45] the authors show how the differential input function can be characterised by a strictly increasing (in space) function $h(x,t)$ solving the backward heat equation and further how $h(x,t)$ can be represented in an integral form with respect to some finite Borel measure. The optimal portfolio and wealth process can then be expressed in terms of h and the market inputs. These results are used in [47] to establish that under time-monotone forward criteria the initial preferences fully characterise the future optimal strategies, given the initial preferences have an integral representation with respect to a finite Borel measure. In Källblad et al. [26] forward performance criteria are combined with ambiguity-averse portfolio selection and the concept of so-called *robust forward criteria* is introduced, which aims to keep the characteristic properties of forward utility functions, whilst being robust with respect to model uncertainty. In a recent extension, Angoshtari et al. [1] introduced a discrete-time forward performance process moving away from the assumption of continuous-time updating of preferences and market assessments, and they provide an algorithm to dynamically construct such *predictable performance processes* in a binomial market model with dynamically updated parameters. Most recently, He et al. [17] aimed to combine the theories of (time-monotone) forward utilities and of rank-dependent utilities - which provide an important alternative to expected utility theory by weakening the independence axiom (the theory of rank-dependent utilities has been introduced by Quiggin, see [51, 52]) - in a consistent manner by combining a forward utility and a distortion process.

While the class of time-monotone performance processes is particularly popular because the expressions are tractable and allow for the computation of explicit solutions, various attempts have been made to give a general characterisation of forward utility functions. Musiela & Zariphopoulou [46] derive an SPDE (see equations (13) and (14) in the next section) that describes the dynamics of a forward performance process that allows an Itô decomposition, and compare it with the SPDE that arises from the value function of the traditional approach. A more rigorous treatment of SPDEs and related SDEs arising in the context of forward utilities is given by El Karoui & Mrad [14]. Shkolnikov et al. [54] examine the asymptotic behaviour of solutions to the SPDE that take a specific form by depending explicitly on non-traded stochastic factors in an incomplete market, and provide explicit formulae for the leading order and first-order correction terms. Nadtochiy and Tehranchi [48] also study the SPDE, which in fact is an analogue to an ill-posed HJB, and in the case when the SPDE can be reduced to a linear parabolic equation they derive integral expressions for the positive solutions. The special class of homothetic forward utilities - which are performance processes with a dependence on wealth in power form - and their characterisation is studied in Nadtochiy & Zariphopoulou [49] and Liang & Zariphopoulou [32]. Since for the classical approach a huge and important field of study is the dual side of the problem, which helps to characterise the value function and solutions to the optimal control problem (see, e.g., Pham [50], Chapter 7), a natural approach to characterise forward performance processes would be to consider their dual side. However, duality has rarely been studied thus far in the forward utility framework. Zitkovic [59] provides a first dual characterisation of forward utilities and establishes necessary and sufficient conditions for exponential-type forward performance processes to be time-consistent. At the same time Berrier et al. [4] independently established necessary and sufficient conditions for a utility process to be a forward utility by imposing conditions on its convex conjugate. More recently, Choulli and Ma [11] took another approach and characterised forward utilities coming from HARA initial conditions via martingale processes by using the concept of Hellinger processes. The theory of forward utilities also found applications not only in optimal investment problems (Zariphopoulou [57] and Musiela & Zariphopoulou [41, 42]), but also in the domain of indifference pricing (Musiela & Zariphopoulou [43], Leung et al. [31], Musiela et al. [40]) as well as risk-measurement (Zitkovic & Zariphopoulou [58] and Chong et al. [9]).

As we will see later, an important quantity in the representation of the optimal strategy is the *local risk-tolerance function*, which for a performance process $u(\cdot,\cdot) \in \mathcal{C}^{2,0}$ is given by

$$R(x,t) = -\frac{u_x(x,t)}{u_{xx}(x,t)}, \qquad t \in \mathcal{T}. \tag{12}$$

The asymptotic behaviour of the local risk tolerance function for large x and t is studied by Geng and Zariphopoulou [15]

4 Optimal investment under forward utilities

In this section we examine the optimal investment strategies that arise from forward performance criteria. We will first introduce a new class of forward utility functions called *Itô-type forward utility processes*, which uses the characterising SPDE derived in [46] and [14] and comprises all forward performance processes that have an Itô decomposition, so in particular this class contains the most widely used classes of performance processes that appeared in the literature so far as special cases. The new definition allows us to get very general results for the optimal investment strategies (Theorem 4.2) and to draw some interesting conclusions regarding the structure of the strategies that arise from forward utilities (Corollaries 4.2.1 and 4.2.2). After deriving the general results, we will consider specific examples of market models and of specific forward utility functions (which have an explicit solution) and examine how their optimal policies coincide or differ from the optimal strategies from the classical framework.

4.1 Classes of forward performance processes

Definition 2. We call a function $U(\cdot,\cdot) : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ an Itô-type forward utility process, if it satisfies

a) U is a forward performance process, i.e. satisfies Definition 1,

b) $U \in \mathcal{C}^{3,0}(\mathbb{R} \times \mathbb{R}_+)$,

c) U has the Itô decomposition

$$dU(x,t) = b(x,t)dt + v(x,t) \cdot dW_t, \tag{13}$$

where $v(\cdot,\cdot) \in \mathcal{C}^{2,0}(\mathbb{R}^m)$ is $\mathbb{F}$-adapted, $b(\cdot,\cdot)$ is given by

$$b(x,t) = \frac{1}{2} \frac{\|U_x(x,t)\lambda_t + \sigma_t^{tr}\sigma_t(\sigma_t^{tr}\sigma_t)^{-1}v_x(x,t)\|^2}{U_{xx}(x,t)}, \qquad t \geq 0, \tag{14}$$

d) U satisfies the conditions given in Appendix A.1, so that the Itô-Ventzell formula[5] can be applied to the process $U(X_t^\pi, t), t \geq 0$, for any admissible wealth process X^π.

Remark 1. We point out that the object $U(\cdot,\cdot)$ in the above definition is a stochastic process itself, even for deterministic inputs (x,t), as is shown by (13). Processes of this form are also known in the literature as *random fields* or *stochastic flows* - for an introduction see Section 1 in El Karoui & Mrad [14] or the book by Kunita [30]. One could also use the notation $U_t(x)$ instead of $U(x,t)$ to emphasise this point.

Remark 2. The expression (14) for the drift has been derived in [46] and more rigorously in [14], who show that any forward performance process which allows an Itô decomposition must have a drift of this form. We introduce this particular class of forward performance processes since it allows us to get very general results for investment strategies that arise from performance processes that have an Itô decomposition. In fact, to the best of our knowledge every forward utility function for which an explicit form has appeared in the literature thus far is a member of this class; in particular the important class of forward performance processes constructed from a differential and stochastic inputs, which is introduced below.

In one of their first papers on forward utilities, Musiela and Zariphopoulou [44] introduce a specific class of forward performance processes which are constructed by combining a deterministic function with stochastic inputs. The authors introduce the processes

$$dN_t = N_t\delta_t(\lambda_t dt + dW_t), \qquad N_0 = 1, \tag{15}$$

$$dZ_t = Z_t\phi_t \cdot dW_t, \qquad Z_0 = 1, \tag{16}$$

$$dA_t = \|\lambda_t + \phi_t - \delta_t\|^2 dt, \qquad A_0 = 0, \tag{17}$$

where $(\delta_t)_{t\geq 0} \in \mathbb{R}^{m\times 1}, (\phi_t)_{t\geq 0} \in \mathbb{R}^{m\times 1}$ are $\mathbb{F}$-adapted, bounded by a deterministic constant[6] and satisfy $\sigma_t^{tr}\sigma_t(\sigma_t^{tr}\sigma_t)^{-1}\delta_t = \delta_t$, $\sigma_t^{tr}\sigma_t(\sigma_t^{tr}\sigma_t)^{-1}\phi_t = \phi_t$, $t \geq 0$, and λ_t is given by (4). The process $A_t = \int_0^t \|\lambda_s\|^2 ds$ is commonly known in the literature as the *mean-variance tradeoff process* and is used here - slightly adapted - to rescale the time argument. N is a numeraire with respect to which the wealth is measured. The authors call the process Z *market view process* since it corresponds to a change of measure and thus allows for flexibility to incorporate "if the investor has different views about the future market movements or faces trading constraints" [44], p. 168. They establish following main characterisation theorem (cf. Theorem 4 in [44]).

[5] For a comprehensive discussion with all technical details regarding the Itô-Ventzell formula, also called *generalised Itô formula*, see Chapter 3 of Kunita [30].

[6] Again, the condition $\int_0^t \|\delta_s\|^2 + \|\phi_s\|^2 ds < \infty$, a.s., $t \in \mathcal{T}$, would be sufficient here.

Theorem 4.1. Let $u(\cdot,\cdot) \in \mathcal{C}^{4,1}(\mathbb{R} \times \mathbb{R}_+)$, such that $u(\cdot,\cdot)$ is concave and increasing in the spatial argument and satisfies the PDE

$$u_t u_{xx} = \frac{1}{2} u_x^2 \tag{18}$$

$$u(x,0) = U(x) \tag{19}$$

for some utility function U. Then the process

$$U_t(x) = u\Big(\frac{x}{N_t}, A_t\Big) Z_t, \qquad t \geq 0, \tag{20}$$

is a forward utility process corresponding to the initial condition $U_0(x) = U(x)$.

Proof. To show that the process (20) indeed satisfies the defining properties of a forward performance process (cf. Definition 1), one needs to apply Itô's Lemma to the process defined in (20) with x being replaced by the wealth process X^π, whose dynamics are given by (9), and use the definitions of the differential and stochastic inputs. A detailed version of the proof can be found in the Appendix of [44]. □

A special case of this class of utility functions is given by the choices $\delta_t = \phi_t = 0$, which means that $N = Z = 1$. One can easily verify that the resulting performance process $U_t(x) = u(x, A_t)$ is monotonically decreasing in time, and thus of finite variation. This implies that the volatility coefficient from the Itô decomposition is given by $v(x,t) = 0$ for all $(x,t) \in \mathbb{R} \times \mathbb{R}_+$. Musiela & Zariphopoulou call this type of performance processes *time-monotone* forward performance processes [45, 47].

Remark 3. By applying Itô's Lemma to the forward performance process defined by (20) (this is done in the proof of Lemma 2, see Equation (46)) one immediately sees that the processes characterised in Theorem 4.1 have an Itô decomposition, and one can easily check that the condition on the drift (14) is satisfied. Thus, these kinds of performance processes are indeed members of the class of Itô-type forward utilities.

The above-described type of forward utilities is particularly popular in the literature (see e.g. [45, 47]), since by specifying classical utility functions as initial conditions one can solve the PDE (18) and compute the differential input functions corresponding to the respective initial conditions. Consequently, one gets an explicit representation for the forward performance process. We will show this below, where we consider the three most widely used utility functions (see, e.g., Gerber and Pafumi [16]) for the initial condition

1. Power utility: $$U(x) = \frac{x^p}{p}, \qquad p < 1, \quad p \neq 0, \quad x \in \mathbb{R}_+, \tag{21}$$
2. Logarithmic utility: $$U(x) = \log(x), \qquad x > 0, \tag{22}$$
3. Exponential utility: $$U(x) = -e^{-\alpha x}, \qquad \alpha > 0, x \in \mathbb{R}. \tag{23}$$

The power and logarithmic utility function belong to the class of *hyperbolic absolute risk aversion* (HARA) utility functions, and the exponential utility function is the most noted example for a *constant absolute risk aversion* (CARA) utility function.

Lemma 1. The differential input functions $u(\cdot,\cdot)$ corresponding to the initial conditions (21)-(23) as well as the respective forward performance processes and their risk tolerance functions are given by

1. Power forward utility process:

$$u(x,t) = \frac{x^p}{p} e^{\frac{q}{2}t}, \qquad (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+, \tag{24}$$

$$U_t(x) = \frac{1}{p}(\frac{x}{N_t})^p e^{\frac{q}{2}A_t} Z_t, \qquad p \in (-\infty, 1)\backslash\{0\}, \ \frac{1}{p} + \frac{1}{q} = 1, \tag{25}$$

$$R(x,t) = \frac{x}{(1-p)}, \tag{26}$$

2. Logarithmic forward utility process:

$$u(x,t) = \log(x) - \frac{t}{2}, \qquad (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+, \tag{27}$$

$$U_t(x) = \Big(\log\big(\frac{x}{N_t}\big) - \frac{1}{2}A_t \Big) Z_t, \tag{28}$$

$$R(x,t) = x, \tag{29}$$

3. Exponential forward utility process:

$$u(x,t) = -e^{-\alpha x + \frac{t}{2}}, \qquad (x,t) \in \mathbb{R} \times \mathbb{R}_+, \tag{30}$$

$$U_t(x) = -\exp\Big(-\alpha\big(\frac{x}{N_t}\big) + \frac{A_t}{2} \Big) Z_t, \qquad \alpha > 0, \tag{31}$$

$$R(x,t) = \frac{N_t}{\alpha}. \tag{32}$$

Proof. These results can be found in [44], but without proof. Hence we give a rigorous proof in Appendix A.2. □

4.2 Optimal investment strategies

The following main result of this section gives the optimal investment strategy for Itô-type forward performance processes in an incomplete Itô-process financial market. The optimal strategy has appeared in the literature as part of the heuristic derivation of the SPDE in Musiela & Zariphopoulou [46] and is also established and proven in El Karoui & Mrad [14], see Theorem 3.1 i). However, the formulation of this result in terms of the newly introduced Itô-type forward utilities, which emphasises the generality of this result, is new.

Theorem 4.2. Let U be an Itô-type forward utility process as in Definition 2. Then the optimal investment strategy is given by

$$\pi_t^* = (\sigma_t \sigma_t^{tr})^{-1} \sigma_t \Big(R(X_t^{\pi^*}, t)\lambda_t - \frac{1}{U_{xx}(X_t^{\pi^*}, t)} v_x(X_t^{\pi^*}, t) \Big), \qquad t \geq 0. \tag{33}$$

Proof. We will give the proof for completeness; it follows the ideas in El Karoui & Mrad [14], Theorem 3.1.

Let U be an Itô-type forward performance process satisfying Definition 2. Then we can apply the Itô-Ventzell formula (see Theorem 3.3.1 in [30]) to derive

$$\begin{aligned} dU(X_t^\pi, t) =& b(X_t^\pi, t)dt + v(X_t^\pi, t) \cdot dW_t + U_x(X_t^\pi, t)dX_t^\pi + \frac{1}{2}U_{xx}(X_t^\pi, t)d\left\langle X^\pi \right\rangle_t \\ &+ v_x^{tr}(X_t^\pi, t)d\left\langle W, X^\pi \right\rangle_t \qquad (34) \\ =& \Big(b(X_t^\pi, t) + U_x(X_t^\pi, t)\pi_t^{tr}\sigma_t\lambda_t + \frac{1}{2}U_{xx}(X_t^\pi, t)(\pi_t^{tr}\sigma_t)(\pi_t^{tr}\sigma_t)^{tr} \\ &+ v_x^{tr}(X_t^\pi, t)\sigma_t^{tr}\pi_t\Big)dt + \Big(U_x(X_t^\pi, t)\pi_t^{tr}\sigma_t + v(X_t^\pi, t)\Big) \cdot dW_t. \qquad (35) \end{aligned}$$

Now we impose the condition that for the optimal π^* the process $\big(U(X_t^{\pi^*}, t)\big)_{t\geq 0}$ must be a martingale. Hence we apply first order conditions

$$\frac{\partial}{\partial \pi}: \qquad U_x(X_t^{\pi^*}, t)\sigma_t\lambda_t + U_{xx}(X_t^{\pi^*}, t)\sigma_t\sigma_t^{tr}\pi^* + \sigma_t v_x(X_t^{\pi^*}, t) \overset{!}{=} 0, \qquad (36)$$

which gives us that the optimal control is given by

$$\pi_t^* = -(\sigma_t\sigma_t^{tr})^{-1}\sigma_t\Big(\frac{U_x(X_t^{\pi^*}, t)\lambda_t}{U_{xx}(X_t^{\pi^*}, t)} + \frac{v_x(X_t^{\pi^*}, t)}{U_{xx}(X_t^{\pi^*}, t)}\Big), \qquad t \geq 0. \qquad (37)$$

Second order condition and the concavity of $U(\cdot, \cdot)$ assert that π_t^* is indeed a supremum. With the definition of the risk tolerance function (12) this establishes the result. □

Corollary 4.2.1. Let U be an Itô-type forward utility process with $v(t, x) = \vartheta(t)$, so the volatility of the forward performance does not depend on the wealth level x. Then the optimal portfolio according to this forward performance criterion is always the myopic portfolio

$$\pi_t^* = (\sigma_t\sigma_t^{tr})^{-1}\sigma_t\lambda_t R(X_t^{\pi^*}, t), \qquad t \geq 0. \qquad (38)$$

We emphasise the generality of this statement, which holds for any Itô-type forward utility process whose volatility process does not depend on the wealth argument, and in a general incomplete Itô-process market. So far in the literature, this observation has only been stated for time-monotone performance criteria in a single stochastic factor model ($d = 1, m = 2$) in [57] and [46].

Corollary 4.2.2. Assume that the stock-price process is driven by only parts of the Brownian motion, and some sources of randomness in the market are external stochastic factors, which, for example, might be driving the volatility and/or market price of risk process. Mathematically, let $B := (W^{(1)}, W^{(2)}, ..., W^{(k)})$, which is a k-dimensional Brownian motion[7], where $d \leq k \leq m$, and assume that the wealth dynamics satisfy

$$dX_t^\pi = \pi_t\sigma_t(\lambda_t dt + dB_t), \qquad (39)$$

[7] Without loss of generality we assume that the sources of randomness driving the stock price are in the first k components of the Brownian motion W.

for $\mathbb{F}$-adapted processes $(\sigma_t)_{t\geq 0} \in \mathbb{R}^{d\times k}$ and $(\lambda_t)_{t\geq 0} \in \mathbb{R}^k$. Then the optimal strategy of an Itô-type forward performance criterion $U(\cdot,\cdot)$ only depends on the components of its volatility process - denoted by $\bar{v}(\cdot,\cdot) \in \mathbb{R}^m$ - which correspond to the sources of randomness which are perfectly correlated with the Brownian motion driving the stock price, by our assumption the first k components.

Proof. Let $0 \leq d \leq k \leq m$. Denote the first k components of the volatility process of the forward performance criterion (cf. Itô decomposition (13)) by $v(\cdot,\cdot)$ and the remaining $m-k$ components by $v^{\perp}(\cdot,\cdot)$, so that $\bar{v} = (v, v^{\perp})^{tr} \in \mathbb{R}^m$. Let $\bar{\lambda} = (\lambda, \mathbf{0}_{m-k})^{tr} \in \mathbb{R}^m$, and $\bar{\sigma} = (\sigma, \mathbf{0}_{d\times m-k}) \in \mathbb{R}^{d\times m}$ be the augmented market price of risk and volatility process respectively. Then by Theorem 4.2 the optimal investment strategy in the forward utility framework with the Itô-type performance process $U(\cdot,\cdot)$ is given by

$$\pi_t^* = (\bar{\sigma}_t\bar{\sigma}_t^{tr})^{-1}\bar{\sigma}_t\Big(R(X_t^{\pi^*},t)\bar{\lambda}_t - \frac{1}{U_{xx}(X_t^{\pi^*},t)}\bar{v}_x(X_t^{\pi^*},t)\Big) \tag{40}$$

$$= (\sigma_t\sigma_t^{tr})^{-1}\sigma_t\Big(R(X_t^{\pi^*},t)\lambda_t - \frac{1}{U_{xx}(X_t^{\pi^*},t)}v_x(X_t^{\pi^*},t)\Big), \tag{41}$$

since in the first line we have that

$$\bar{\sigma}_t\bar{v}_x(X_t^{\pi^*},t) = \big(\sigma_t, \mathbf{0}_{d\times m-k}\big)\big(v_x(X_t^{\pi^*},t), v_x^{\perp}(X_t^{\pi^*},t)\big)^{tr} = \big(\sigma_t v_x(X_t^{\pi^*},t), \mathbf{0}_{d\times m-k}\big), \tag{42}$$

and analogously for $\bar{\sigma}_t\bar{\lambda}_t$, which proves the claim. □

Remark 4. Corollary 4.2.2 implies that unhedgeable risk is ignored in the optimal strategies from Itô-type forward performance processes, since the coefficients of the stochastic factors which cannot be perfectly replicated have no impact on the optimal strategy. Only if the external stochastic factor is included into the components of the volatility process driving the stock price, e.g. by writing $v(X_t, Y_t, t)$, one can make the optimal strategy dependent on the external stochastic factor Y. This is an important insight for risk managers, if they want to include some extra hedging demand into their risk management strategy to account for the external risk factors.

In order to get a better understanding of the implications of the result of Theorem 4.2 and the corollaries, we will subsequently consider the optimal strategies for the forward performance processes characterised in Theorem 4.1, which form a sub-class of Itô-type forward processes, as we noted in Remark 3. The explicit expression for the risk tolerance function of some important members of this type of forward utilities (cf. Lemma 1) allows for comparison of their optimal strategies with their classical counterparts, which will be done in Section 4.3. The next Lemma gives the optimal strategy for this type of forward utilities, a result which has already been formulated by Musiela and Zariphopoulou in [44, 46] (Theorem 8 and Example 6.2.3 respectively). However, the proof given below shows how these results follow easily from the more general result of Theorem 4.1 established herein.

Lemma 2. Let U be a forward performance process as characterised by Theorem 4.1. Then the corresponding optimal investment strategy is given by

$$\pi_t^* = (\sigma_t\sigma_t^{tr})^{-1}\sigma_t\Big(R(X_t^{\pi^*},t)(\lambda_t + \phi_t - \delta_t) + \delta_t X_t^*\Big), \qquad t \geq 0, \tag{43}$$

where we recall that the risk-tolerance function $R(\cdot,\cdot)$ is given by

$$R(X_t^{\pi^*},t) = -\frac{\frac{\partial}{\partial x}U_t(X_t^*)}{\frac{\partial^2}{\partial x^2}U_t(X_t^{\pi^*})} = -\frac{u_x\big(\frac{X_t^{\pi^*}}{N_t},A_t\big)Z_tN_t}{u_{xx}\big(\frac{X_t^{\pi^*}}{N_t},A_t\big)Z_t}. \tag{44}$$

Proof. Since the differential input is by assumption smooth enough, we can apply Itô's formula to compute

$$dU_t(x) = d\Big(u\big(\frac{x}{N_t},A_t\big)Z_t\Big) = Z_t d\Big(u\big(\frac{x}{N_t},A_t\big)\Big) + u\big(\frac{x}{N_t},A_t\big)dZ_t + d\left\langle u\big(\frac{x}{N},A\big),Z\right\rangle_t \tag{45}$$

and standard computations give that

$$\begin{aligned} dU_t(x) =& Z_t\Big(u_t\big(\frac{x}{N_t},A_t\big)\|\lambda_t+\phi_t-\delta_t\|^2 + u_x\big(\frac{x}{N_t},A_t\big)\frac{x}{N_t}\delta_t^{tr}(\delta_t-\lambda_t-\phi_t) \\ &+\frac{1}{2}u_{xx}\big(\frac{x}{N_t},A_t\big)\frac{x^2}{N_t^2}\delta_t^{tr}\delta_t\Big)dt + Z_t\Big(u\big(\frac{x}{N_t},A_t\big)\phi_t - u_x\big(\frac{x}{N_t},A_t\big)\frac{x}{N_t}\delta_t\Big)\cdot dW_t. \end{aligned} \tag{46}$$

We observe that the volatility vector of $U_t(\cdot)$ is given by

$$v(x,t) = Z_t\Big(u\big(\frac{x}{N_t},A_t\big)\phi_t - u_x\big(\frac{x}{N_t},A_t\big)\frac{x}{N_t}\delta_t\Big) \tag{47}$$

and thus we compute

$$v_x(x,t) = Z_t\Big(u_x\big(\frac{x}{N_t},A_t\big)\frac{1}{N_t}(\phi_t-\delta_t) - u_{xx}\big(\frac{x}{N_t},A_t\big)\frac{x}{N_t^2}\delta_t\Big) \tag{48}$$

$$= \frac{\partial}{\partial x}U_t(x)(\phi_t-\delta_t) - \frac{\partial^2}{\partial x^2}U_t(x)x\delta_t. \tag{49}$$

We plug this into the result from Theorem 4.2 and verify that

$$\pi_t^* = (\sigma_t\sigma_t^{tr})^{-1}\sigma_t\Big(R(X_t^*,t)\lambda_t - \frac{\frac{\partial}{\partial x}U_t(X_t^{\pi^*})}{\frac{\partial^2}{\partial x^2}U_t(X_t^{\pi^*})}(\phi_t-\delta_t) + \frac{\frac{\partial^2}{\partial x^2}U_t(X_t^{\pi^*})}{\frac{\partial^2}{\partial x^2}U_t(X_t^{\pi^*})}X_t^{\pi^*}\delta_t\Big) \tag{50}$$

$$= (\sigma_t\sigma_t^{tr})^{-1}\sigma_t\Big(R(X_t^*,t)(\lambda_t+\phi_t-\delta_t) + X_t^{\pi^*}\delta_t\Big). \tag{51}$$

□

Remark 5. The special case of time-monotone forward preferences is given by the choices $\delta_t = \phi_t = 0$, which leads to $v(x,t) = 0$ in the Itô representation of U (cf. Equation (46)). Thus Lemma 2 certifies that for these performance processes the optimal strategy will always be the myopic portfolio[8].

[8]This is also a direct consequence of Corollary 4.2.1.

4.3 Examples and comparison of classical versus forward optimal strategies

We now consider some examples of standard market models and compare the optimal strategies from the classical approach and the forward approach. In the classical approach the typical procedure to obtain the optimal investment strategy is as follows (see, e.g., Chapter 3 in Pham [50] for a rigorous treatment of the heuristically explained procedure outlined herein). First, one writes down the market model (in particular the wealth dynamics) and defines the value function. Then one applies the dynamic programming principle to derive the HJB equation satisfied by the value function. Since the investment horizon and the utility function are a priori fixed, one obtains a terminal condition for the HJB and can thus - in some market models - solve it explicitly to get the value function, which then allows one to deduce the optimal strategy.

In the forward utility framework, the computation of the optimal strategy typically involves following three steps: In the first step, one writes down the market model, most importantly the wealth dynamics. The next step involves specifying the forward utility process satisfying conditions (i)-(iv) in Definition 1. By Theorem 4.1 one way to do this is by specifying a differential as well as stochastic inputs. In the last step one computes the dynamics of the forward performance process and derives the optimal strategy π^* by imposing the condition that the forward utility process is a martingale.

In order to enable comparability between the two approaches, we take the utility function which serves as a terminal condition in the traditional approach and impose it as initial condition of the corresponding forward performance process. We will focus on the performance criteria constructed from a differential and stochastic inputs, and specifically the ones given by Lemma 1, as they allow for explicit solutions.

Example 1: Black Scholes market model
The *Black-Scholes market model*[9] consists of one risk-free and one risky asset, modelled as in (1) with $\mu_t = \mu$, $\sigma_t = \sigma$ being constants. We have that $d = m = 1$, therefore the market is complete. The wealth process follows

$$dX_t^\pi = \pi_t \sigma(\lambda dt + dW_t), \qquad X_0^\pi = x. \tag{52}$$

Classical approach We assume that the investor fixes an investment horizon T and a utility function $U(\cdot)$. In the classical case with constant model parameters this problem is often referred to as *Merton problem.* It is well-studied and serves as a popular example to introduce stochastic control theory for finance (see, e.g., Rogers [53], Chapter 1, or Pham [50], Chapters 2 and 3). Standard arguments which can be found in aforementioned references show that the optimal control in feedback form is given by

$$\pi_t^* = -\frac{\lambda}{\sigma}\frac{u_x(x,t)}{u_{xx}(x,t)} = \frac{\lambda}{\sigma}R(x,t), \qquad 0 \leq t \leq T, \tag{53}$$

where $u(\cdot,\cdot)$ is the value function of the optimal control problem. This implies that for the 3 different terminal utility functions (21)-(23) we obtain that the optimal policies are

[9] Also known as Black-Scholes-Merton market model.

given by (the results can be found, e.g., in Monoyios [39])

1. Power utility: $$\pi_t^* = \frac{\lambda}{\sigma}\frac{X_t^{\pi^*}}{(1-p)}, \qquad 0 \leq t \leq T, \tag{54}$$

2. Logarithmic utility: $$\pi_t^* = \frac{\lambda}{\sigma}X_t^{\pi^*}, \qquad 0 \leq t \leq T, \tag{55}$$

3. Exponential utility: $$\pi_t^* = \frac{\lambda}{\sigma\alpha}, \qquad 0 \leq t \leq T. \tag{56}$$

The optimal policy is to put a constant fraction of wealth into the risky asset in the case of power and logarithmic utility ($\theta_t = \frac{\lambda}{\sigma(1-p)}$ and $\theta_t = \frac{\lambda}{\sigma}$ respectively) and to invest a constant amount in the stock in the case of exponential preferences. We identify these policies as the myopic portfolios.

Forward utility approach Using the risk tolerance functions of the time-monotone forward utilities from Lemma 1, the results from Lemma 2 in the one-dimensional case simplify to

1. Power forward utility: $$\pi_t^* = \frac{X_t^{\pi^*}}{(1-p)}\frac{\big(\lambda + \phi_t - \delta_t\big)}{\sigma} + X_t^{\pi^*}\frac{\delta_t}{\sigma}, \qquad t \geq 0, \tag{57}$$

2. Logarithmic forward utility: $$\pi_t^* = X_t^{\pi^*}\frac{\big(\lambda + \phi_t\big)}{\sigma}, \qquad t \geq 0, \tag{58}$$

3. Exponential forward utility: $$\pi_t^* = N_t\frac{\big(\lambda + \phi_t - \delta_t\big)}{\sigma\alpha} + X_t^{\pi^*}\frac{\delta_t}{\sigma}, \qquad t \geq 0. \tag{59}$$

Remark 6. We observe that in the case of constant model parameters the traditional and the classical approach yield the same optimal policies if we use a time-monotone forward performance process, for which $\phi_t = \delta_t = 0$. Since we know from classical theory that a logarithmic performance criterion gives the myopic portfolio in a general incomplete Itô-process market (see, e.g., Example 7.2 in Karatzas & Shreve [29]), the classical optimal strategies can - for the logarithmic case - always be replicated by time-monotone performance criteria.

Example 2: Stochastic factor model

Next, we consider an incomplete market model with one risky, one risk-free asset and a stochastic factor driving the volatility and market price of risk process, so $m = 2, d = 1$. The generic model is given by

$$dS_t = S_t\mu(Y_t)dt + S_t\sigma(Y_t)dB_t \tag{60}$$

$$dY_t = a(Y_t)dt + b(Y_t)d\tilde{B}_t, \tag{61}$$

where $\mu(\cdot), \sigma(\cdot), a(\cdot), b(\cdot)$ are one-dimensional, adapted processes such that (60) and (61) admit a strong solution. B and $\tilde{B}$ are one-dimensional Brownian motions with correlation ρ, so we can rewrite the Y-dynamics as

$$dY_t = a(Y_t)dt + b(Y_t)(\rho dB_t + \bar{\rho}dB_t^{\perp}), \qquad \bar{\rho} = \sqrt{1-\rho^2}, \quad Corr(B, B^{\perp}) = 0. \tag{62}$$

The wealth dynamics are then given by

$$dX_t^{\pi} = \pi_t\sigma(Y_t)\big(\lambda(Y_t)dt + dB_t\big), \tag{63}$$

where $\lambda(Y_t) = \frac{\mu(Y_t)}{\sigma(Y_t)}$ is the (one-dimensional) market price of risk.

Classical approach As before, we assume that the investor chooses both a time horizon and a terminal utility function and then aims to maximise expected terminal utility. Zariphopoulou [57] shows that the optimal portfolio in feedback form is given by

$$\pi_t^* = -\frac{\lambda(Y_t)}{\sigma(Y_t)}\frac{u_x}{u_{xx}} - \rho\frac{b(Y_t)}{\sigma(Y_t)}\frac{u_{xy}}{u_{xx}}, \qquad 0 \le t \le T. \tag{64}$$

Solving the corresponding HJB equation for the 3 different terminal utility functions yields (see Zariphopoulou [56] for a detailed derivation of the power-case, an analogue approach yields the results for the other 2 cases, which can be found in [57], Section 3.1)

1. Power utility: $$\pi_t^* = \frac{\lambda(Y_t)}{\sigma(Y_t)}\frac{X_t^{\pi^*}}{(1-p)} + \frac{\rho}{(1-q\rho^2)}\frac{1}{(1-p)}\frac{b(Y_t)}{\sigma(Y_t)}\frac{f_y(t,y)}{f(t,y)}X_t^{\pi^*}, \tag{65}$$
2. Logarithmic utility: $$\pi_t^* = \frac{\lambda(Y_t)}{\sigma(Y_t)}X_t^{\pi^*}, \qquad 0 \le t \le T, \tag{66}$$
3. Exponential utility: $$\pi_t^* = \frac{\lambda(Y_t)}{\sigma(Y_t)\alpha} + \frac{\rho}{\bar{\rho}^2}\frac{b(Y_t)}{\sigma(Y_t)\alpha}\frac{h_y(t,Y_t)}{h(t,Y_t)}, \qquad 0 \le t \le T. \tag{67}$$

for
$$f(t,y) := \mathbb{E}^{\tilde{P}_1}[\exp\Big(-\frac{1}{2}q(1-q\rho^2)\int_t^T \lambda(Y_s)^2 ds\Big)|Y_t = y], \tag{68}$$
$$h(t,y) := \mathbb{E}^{\tilde{P}_2}[\exp\Big(-\frac{\bar{\rho}^2}{2}\int_t^T \lambda(Y_s)^2 ds\Big)|Y_t = y], \tag{69}$$

where $\tilde{\mathbb{P}}_1$, $\tilde{\mathbb{P}}_2$ are measures on $(\Omega, \mathcal{F})$ equivalent to $\mathbb{P}$ and such that Y has dynamics under $\tilde{\mathbb{P}}_1$ and $\tilde{\mathbb{P}}_2$ respectively

$$dY_t = \big(a(y) - \rho q b(y)\lambda(y)\big)dt + b(y)dW_t^{\tilde{\mathbb{P}}_1} \tag{70}$$
$$dY_t = \big(a(y) - \rho b(y)\lambda(y)\big)dt + b(y)dW_t^{\tilde{\mathbb{P}}_2}. \tag{71}$$

Forward utility approach The introduction of the stochastic factor does not change the structure of the wealth dynamics, which implies that we essentially get the same results as in the Black-Scholes model (Example 1). However, we need to be a bit careful and cannot immediately use the results from the previous example, because looking back to the general optimal strategy of forward performance processes, we see that it is given in terms of vector-valued (here $m = 2$) coefficients, whereas the wealth dynamic in (63) is only one-dimensional. Hence we rewrite (63) as

$$dX_t^\pi = \pi_t\bar{\sigma}_t(\bar{\lambda}_t dt + dW_t), \tag{72}$$

for $\bar{\sigma}_t = (\sigma(Y_t), 0)^{tr}$, $\bar{\lambda}_t = (\lambda(Y_t), 0)^{tr}$. Then we can reuse the results from the previous example with the coefficients $\lambda, \sigma, \phi_t, \delta_t$ replaced by the corresponding 2-dimensional processes (which we will denote with a bar, e.g. $\bar{\phi}_t = (\phi_t^1, \phi_t^2)^{tr}$).

1. Power f. utility:

$$\pi_t^* = (\bar{\sigma}_t \bar{\sigma}_t{}^{tr})^{-1} \bar{\sigma}_t \Big(\frac{X_t^{\pi^*}}{(1-p)} (\bar{\lambda}_t + \bar{\phi}_t - \bar{\delta}_t) + \bar{\delta}_t X_t^{\pi^*} \Big) \tag{73}$$
$$= \frac{X_t^{\pi^*}}{1-p} \frac{\big(\lambda(Y_t) + \phi_t^1 - \delta_t^1\big)}{\sigma(Y_t)} + X_t^{\pi^*} \frac{\delta_t^1}{\sigma(Y_t)}, \qquad t \geq 0,$$

2. Logarithmic f. utility:

$$\pi_t^* = (\bar{\sigma}_t \bar{\sigma}_t{}^{tr})^{-1} \bar{\sigma}_t \Big(X_t^{\pi^*} (\bar{\lambda}_t + \bar{\phi}_t - \bar{\delta}_t) + \bar{\delta}_t X_t^{\pi^*} \Big) \tag{74}$$
$$= X_t^{\pi^*} \frac{\big(\lambda(Y_t) + \phi_t^1\big)}{\sigma(Y_t)}, \qquad t \geq 0,$$

3. Exponential f. utility:

$$\pi_t^* = (\bar{\sigma}_t \bar{\sigma}_t{}^{tr})^{-1} \bar{\sigma}_t \Big(\frac{N_t}{\alpha} (\bar{\lambda}_t + \bar{\phi}_t - \bar{\delta}_t) + \bar{\delta}_t X_t^{\pi^*} \Big) \tag{75}$$
$$= N_t \frac{\big(\lambda(Y_t) + \phi_t^1 - \delta_t^1\big)}{\sigma(Y_t)\alpha} + X_t^{\pi^*} \frac{\delta_t^1}{\sigma(Y_t)}, \qquad t \geq 0.$$

Remark 7. We observe that - as suggested by Corollary 4.2.2 - the optimal strategy only depends on the first component of the volatility vectors of the market view and benchmark process respectively; the unhedgeable source of risk from the stochastic factor is ignored.

Remark 8. Let us assume that the numeraire is given by the risk-free asset, i.e. $\delta = 0, N = 1$. Then, for the volatility process $(\phi_t)_{t\geq 0}$ of the market view process $(Z_t)_{t\geq 0}$ there is - besides some minor integrability and adaptedness requirements - great flexibility for the agent to choose this process according to her beliefs and/or preferences. An open question is how the agent chooses this process. One approach to answer this question would be to work empirically, to look at a past trading period and an investor's investment decisions within this period. One can then try to derive an agent's implicitly chosen market view process, assuming that her decisions were indeed optimal. An investigation in this direction or another account for the interpretation of the market view process (or equivalently for its volatility process $(\phi)_{t\geq 0}$, which uniquely determines the market view process) is left to future research.

Remark 9. We try to recover the optimal strategies of the classical approach by specific choices of the market view process. We observe that in the classical framework, the optimal strategy is given by the myopic portfolio plus a correction term (cf. (65), (67)) - which is often referred to as *excess hedging demand* - and which depends on the volatility process $b(\cdot)$ of the stochastic factor. Hence, the forward approach can only replicate the classical strategy if we incorporate the volatility process of the stochastic factor into the volatility of the forward performance process, and thereby introduce an implicit dependence of the forward performance process on the stochastic factor. Otherwise, the unhedgeable risk from the stochastic factor would be ignored, as noted by Remark 7.

We note following special cases for the choice of the market view process, where we again assume $\delta_t = 0$:

1. $\phi_t^1 = 0$: In case (the first component of) the volatility process is chosen to be 0, we get from the dynamics (46) of the forward performance process that the volatility process corresponding to the correlated Brownian term (the first component) is zero

and we end up with only the myopic portfolio for each of the 3 performance criteria. Thus, any excess hedging demand disappears for forward utility processes with zero volatility (at least in the first component), so in particular for time-monotone ones.

2. $\phi_t^1 = \frac{\rho}{(1-q\rho^2)} b(Y_t) \frac{f_y(A_t,Y_t)}{f(A_t,Y_t)}$, for f as in (65), then with power utility initial condition we recover the same strategy as in the classical approach (65).

3. $\phi_t^1 = \frac{\rho}{\bar{\rho}^2} b(Y_t) \frac{h_y(A_t,Y_t)}{h(A_t,Y_t)}$, where h is given by (67). Analogue to the previous case, we recover the optimal strategy from the classical approach for the exponential performance process.

4. $\phi_t^1 = -\lambda(Y_t)$: This choice leads to $\pi_t^* = 0$, i.e. the agent puts all of her wealth in the risk-free asset, which holds for all 3 forward utility processes.

Example 3: Black Scholes model with drift uncertainty
We observed in the previous two examples that the forward investment strategies stay the same as long as the wealth dynamics do not change and the market coefficients are adapted to the filtration which the investor observes. For example, in the Black-Scholes model it is assumed that the investor can observe the Brownian motion W and hence has full information about the (constant) drift μ of the asset. However, a more realistic assumption is that the agent only observes the price process S, and so the market price of risk λ is unknown. In this example we give a short overview of how one can deal with this situation and by means of filtering theory recover a full-information setup, which then allows us to apply the results from the previous sections. A comprehensive discussion of filtering in the light of partial information models and a detailed derivation of the model used in this section can be found in Monoyios [37].

We consider the case of a Black-Scholes market model, but with the market price of risk being an unknown constant. We assume it to be an $\mathcal{F}_0$-measurable random variable with a Gaussian distribution ($\lambda \sim N(\lambda_0, v_0)$). We can infer the value of the volatility σ from the quadratic variation of the observation process: $\sigma_t^2 = \frac{1}{S_t^2 t} \langle S \rangle_t, 0 \leq t \leq T$. We call the filtration generated by the stock price process $\hat{\mathbb{F}} := (\sigma(S_u : 0 \leq u \leq t))_{0 \leq t \leq T}$ the *observation filtration*. Now the idea is to define an $\hat{\mathbb{F}}$-adapted process $(\hat{\lambda}_t)_{0 \leq t \leq T}$ which describes our best estimate of the market price of risk given the current information. This is a classical filtering problem, and a widely used approach to get the best estimate for λ is given by the *Kalman-Bucy filter* [7], which says that the optimal filter $\hat{\lambda}_t := \mathbb{E}[\lambda | \hat{\mathcal{F}}_t]$ is given by

$$\hat{\lambda}_t = \lambda_0 + \int_0^t v_s d\hat{B}_s, \qquad 0 \leq t \leq T, \tag{76}$$

where $\hat{B}$ is a $\hat{\mathbb{F}}$-standard Brownian motion and v_t is the conditional variance which satisfies

$$dv_t = -v_t^2 dt, \qquad v_0 = v_0, \tag{77}$$

i.e.

$$v_t = \frac{v_0}{1 + v_0 t}, \qquad 0 \leq t \leq T. \tag{78}$$

We observe that $\hat{\lambda}$ is a Gaussian process adapted to $\hat{\mathbb{F}}$. After filtering, we essentially end up with a full information model, where the stock price and wealth process respectively have dynamics

$$dS_t = \sigma S_t(\hat{\lambda}_t dt + d\hat{B}_t) \tag{79}$$

$$dX_t^\pi = \sigma \pi_t(\hat{\lambda}_t dt + d\hat{B}_t), \qquad X_0^\pi = x. \tag{80}$$

Note that our investment strategy must be adapted to the observation filtration, hence $(\pi_t)_{0\le t\le T}$ is an $\hat{\mathbb{F}}$-adapted process.

Classical approach We consider a Merton-type problem with the above-mentioned model, hence we fix a time horizon T and a utility function U for the terminal time. In order to solve the optimal investment problem with the Gaussian drift process, we use duality theory, in particular Theorem 4.3. For a good introduction to duality theory for optimal investment problems we refer the reader to Karatzas et al. [28], or the textbook treatments by Karatzas and Shreve [29], Chapter 6.5, and Pham [50], Chapter 7.

Let the primal problem be defined by

$$u(x) := \sup_{\pi\in\mathcal{A}} \mathbb{E}[U(X_T^\pi)|\hat{\mathcal{F}}_0] \tag{81}$$

$$\text{s.t. } \mathbb{E}[Z_T^{\mathbb{Q}} X_T|\hat{\mathcal{F}}_0] = x, \tag{82}$$

where $Z_t^{\mathbb{Q}} = \mathcal{E}(-\hat{\lambda}\cdot\hat{B})_t$ is the Radon-Nikodym derivative of the (unique) martingale measure $\mathbb{Q}$ with respect to $\mathbb{P}$ on $\hat{\mathcal{F}}_t$. Denote by $\tilde{U}$ the convex conjugate and by I the marginal inverse of the utility function U, which are defined by

$$\tilde{U}(\eta) := \sup_{x\in\mathbb{R}^+} \{U(x) - x\eta\}, \tag{83}$$

$$I(\eta) := U'(\eta)^{-1}. \tag{84}$$

Then the dual value function $\tilde{u}$ is defined by

$$\tilde{u}(\eta) := \mathbb{E}[\tilde{U}(\eta Z_T)|\hat{\mathcal{F}}_0]. \tag{85}$$

Theorem 4.3. (Monoyios [37], Theorem 6) Let the primal and dual value function be given by (81) and (85) respectively.

1. Then u and $\tilde{u}$ are conjugate, i.e.

$$u(x) = \inf_{\eta\in\mathbb{R}^+} \{\tilde{u}(\eta) + x\eta\}, \qquad \tilde{u}(\eta) = \sup_{x\in\mathbb{R}^+} \{u(x) - x\eta\}, \tag{86}$$

which implies that $u'(x) = \eta$ and $\tilde{u}'(\eta) = -x$.

2. The optimal terminal wealth of the primal problem (81) is given by

$$X_T^* = I(\eta Z_T). \tag{87}$$

Proof. We refer to Monoyios [37], Theorem 6. □

Returning to our original optimal investment problem, we proceed in 3 steps to compute the optimal strategy $(\pi_t^*)_{0\le t\le T}$: First, we find the dual value function using (85) and apply Theorem 4.3 to obtain the primal value function. Then, we invoke the second part of Theorem 4.3 to deduce the optimal terminal wealth and impose the martingale condition to obtain the corresponding wealth process $(X_t^*)_{0\leq t\leq T}$. In the last step, we compute the dynamics of the optimal wealth process and compare with (80) to find the corresponding optimal strategy.

We carry out these steps for our 3 standard examples of utility functions and get following results. A detailed derivation of the power utility case can be found in Monoyios [37], Theorem 9, and of the exponential utility case in Björk et al. [6], Proposition 3.8.

1. Power utility: $$\pi_t^* = \frac{\hat{\lambda}_t}{\sigma}\frac{X_t^{\pi^*}}{(1-p)}\Big(\frac{1}{1+qv_t(T-t)}\Big), \qquad 0\leq t\leq T. \tag{88}$$
2. Logarithmic utility: $$\pi_t^* = \frac{\hat{\lambda}_t}{\sigma}X_t^{\pi^*}, \qquad 0\leq t\leq T. \tag{89}$$
3. Exponential utility: $$\pi_t^* = \frac{\hat{\lambda}_t}{\sigma\alpha} - \frac{\sigma_t^{(E)}}{2\sigma\alpha}, \qquad 0\leq t\leq T, \tag{90}$$

where $\sigma_t^{(E)}$ is the variance of the process $E_t := \mathbb{E}[\int_t^T \|\lambda_s\|^2 ds|\hat{\mathcal{F}}_t]$.

Forward utility approach After filtering, we again obtain a full-information model where the wealth dynamics are given by (80). We observe that we are essentially in the same situation as in Example 1, with the only difference being that the constant λ is replaced by the Gaussian process $(\hat{\lambda}_t)_{t\geq 0}$ and we change the filtration $\mathbb{F}\to\hat{\mathbb{F}}$. Hence, we can refer to the previously obtained results, but with $\lambda\to\hat{\lambda}_t$, and deduce that the optimal strategies are given by

1. Power forward utility: $$\pi_t^* = \frac{X_t^{\pi^*}}{(1-p)}\frac{\big(\hat{\lambda}_t+\phi_t-\delta_t\big)}{\sigma} + X_t^{\pi^*}\frac{\delta_t}{\sigma}, \qquad t\geq 0, \tag{91}$$
2. Logarithmic forward utility: $$\pi_t^* = X_t^{\pi^*}\frac{\big(\hat{\lambda}_t+\phi_t\big)}{\sigma}, \qquad t\geq 0, \tag{92}$$
3. Exponential forward utility: $$\pi_t^* = N_t\frac{\big(\hat{\lambda}_t+\phi_t-\delta_t\big)}{\sigma\alpha} + X_t^{\pi^*}\frac{\delta_t}{\sigma}, \qquad t\geq 0. \tag{93}$$

Remark 10. Comparing the optimal strategies of the classical and the forward framework, we observe that for the case of a power utility criterion the forward framework cannot recover the optimal strategy from the classical framework without introducing the time horizon T. In the classical approach, in the limit $t\to T$ the optimal portfolio converges to the myopic strategy (since v_t is bounded). For $t<T$, we note that for $v_0\geq 0$ also $v_t>0$ for all $0\leq t\leq T$ and hence

$$\kappa_t := \Big(\frac{1}{1+qv_t(T-t)}\Big) = \begin{cases} >1 & \text{for } p\in(0,1), \\ <1 & \text{for } p\in(-\infty,0). \end{cases} \tag{94}$$

Thus, the classical Merton strategy is adapted by reducing $(p<0)$ or increasing $(p>0)$ the amount invested in the risky asset by the factor κ_t. The more risk-averse the agent is, the smaller the correction factor, and hence the more the position in the risky asset is reduced.

On the contrary, a time-monotone forward performance criterion would yield an optimal strategy that ignores the parameter uncertainty and just uses the best estimate of the market price of risk for the optimal strategy. Of course, for a non-zero volatility forward performance process the agent could specify the market view process to get a correction term for the position in the risky asset. However, due to the lack of understanding of how the agent chooses this process (see Remark 9), it is unclear if the agent would update her market view process in the case when the assumption on the underlying market model changes (i.e. from known parameters to a model with estimated parameters which includes uncertainty). Since we have established (cf. Corollary 4.2.2) that unhedgeable risk factors are ignored, we conjecture that the agent only changes her market view process if the market environment itself changes, which would imply that the only thing that changes in the optimal strategy is a replacement of the constant market price of risk by the best estimate based on the Kalman-Bucy filter. This however would mean that risks coming from parameter uncertainty remain unaddressed. Future research on the market view process could investigate this hypothesis.

For the exponential utility, the standard strategy is adapted by reducing the amount of money in the stock. We observe that the higher the volatility of the risk-neutral conditional expectation of the mean-variance tradeoff process E, the more we reduce our position in the risky asset in the classical case. We can recover this strategy in the forward approach by choosing the market view volatility process as $\phi_t = -\frac{1}{2}\sigma_t^{(E)}$ and $\delta_t = 0$.

5 Optimal strategies for investment and consumption

So far we have only considered situations where the agent obtains utility only from terminal wealth. However, in a more realistic approach we should take into account that the agent can also use parts of her wealth to consume throughout the investment period and will derive utility from this consumption[10]. The consumption, of course, reduces the wealth of the agent. Hence if we take again our general Itô-process market model (cf. Section 2) the dynamics of the (controlled) wealth process of the agent are now given by

$$dX_t^{\pi,c} = \pi_t\sigma_t(\lambda_t dt + dW_t) - c_t dt \qquad X_0^{\pi,c} = x. \tag{95}$$

We observe that the investor can now control 2 processes, namely the amount of wealth in the risky asset $(\pi_t)_{0\leq t\leq T}$ as well as the $\mathbb{F}$-adapted, non-negative consumption process $(c_t)_{0\leq t\leq T}$ satisfying $\int_0^T c_t dt < \infty$, almost surely. The admissible set contains all pairs c, π such that the same assumptions for an admissible π as outlined below equation (10) hold, and in addition $(c_t)_{0\leq t\leq T}$ from above is such that $0 \leq c_t \leq X_t$ for all $0 \leq t \leq T$.

We will first review how this problem is handled in the classical framework, and then show how these ideas informed Berrier & Tehranchi [5] to define a forward performance criterion for investment and consumption.

[10] In his seminal papers, from which the classical theory of optimising investment by maximising expected utility originated [33, 34], Merton in fact already included consumption in the problem.

5.1 Classical approach

We fix a time horizon $T \in (0, \infty)$ as well as an impatience factor δ[11]. Then the objective is to maximise expected utility from consumption and from terminal wealth over the set of admissible investment strategies and consumption paths

$$\mathbb{E}[\int_0^T e^{-\delta t} U^{(1)}(c_t)dt + U^{(2)}(X_T^{\pi,c})] \to \max!, \tag{96}$$

where $U^{(1)}, U^{(2)}$ are both a priori fixed utility functions. Clearly, how much the agent can consume depends on her level of wealth, hence the value function will be a function of wealth and time. It is given by

$$u(x,t) = \sup_{\pi,c\in\mathcal{A}} \mathbb{E}[\int_t^T e^{-\delta s} U^{(1)}(c_s)ds + U^{(2)}(X_T^{\pi,c})|X_t^{\pi,c} = x]. \tag{97}$$

Standard stochastic control theory (see, e.g., Chapter 3.3 in Pham [50]) shows that the process

$$\zeta_t^{\pi,c} = \int_0^t e^{-\delta s} U^{(1)}(c_s)ds + u(X_t^{\pi,c}, t), \qquad 0 \le t \le T, \tag{98}$$

is a supermartingale for all $(\pi, c) \in \mathcal{A}$ and a martingale for the optimal controls π^*, c^*. Applying first order conditions to the HJB (see, e.g., Chapter 1.2 in Rogers [53] or the proof of Theorem 1 in Merton [34][12]) shows that the optimal controls are given by

$$\pi_t^* = -(\sigma_t \sigma_t^{tr})^{-1}\sigma_t\Big(\frac{u_x(X_t^*, t)\lambda_t}{u_{xx}(X_t^*, t)}\Big), \qquad 0 \le t \le T, \tag{99}$$

$$c_t^* = I^{(1)}\big(e^{\delta t} u_x(X_t^*, t)\big), \qquad 0 \le t \le T, \tag{100}$$

where $X^* = X^{\pi^*,c^*}$ and $I^{(1)}$ denotes the marginal inverse of the consumption utility function $U^{(1)}$.

If there is no prespecified time horizon, then we only choose a utility function of consumption $U^{(1)}$ and a discount factor δ and aim to maximise the expected utility from consumption from now up to an infinite time horizon. Therefore, the objective is

$$\mathbb{E}[\int_0^\infty e^{-\delta t} U^{(1)}(c_t)dt] \to \max! \tag{101}$$

Problem (101) is called the *infinite horizon problem.* While in the finite horizon problem the value function was assumed to be a function of wealth and time, in an infinite horizon problem the lack of a termination time implies that the value function is not time-dependent, as was already noted by Merton [33] when first introducing this problem. Intuitively speaking, the agent is faced with the same problem if she has a wealth of

[11] The impatience factor takes into account that a typical agent would rather consume the same amount today than at any future point in time. Hence, δ serves as a discount coefficient which determines how much less utility an agent derives from future compared to immediate consumption.

[12] Merton's value function also depends on the current asset price, which we don't assume, since it is incorporated in the wealth process. Hence setting the P-derivative to 0 in (27) in [34] gives the same result as we have here.

x at time t_1 as if she had the same wealth at some time $t_2 > t_1$. Since for both points in time the time to maturity is infinite, the agent will adopt the same strategy for a certain level of wealth, regardless at what point in time she hits that level. Therefore, the value function is given by

$$J(x) = \sup_{\pi,c\in\mathcal{A}} \mathbb{E}[\int_0^\infty e^{-\delta t} U^{(1)}(c_t)dt], \tag{102}$$

where without loss of generality we set the current time to zero. The same arguments as in the finite horizon case assert that the process ζ defined in (98) with $u(X_t^{\pi,c}, t)$ replaced by $J(X_t^{\pi,c})$ is a supermartingale for all $\pi, c \in \mathcal{A}$ and a martingale for the optimal π^*, c^*, which have the feedback form

$$\pi_t^* = -(\sigma_t \sigma_t^{tr})^{-1} \sigma_t \Big(\frac{J'(X_t^*)\lambda_t}{J''(X_t^*)} \Big), \qquad t \geq 0, \tag{103}$$

$$c_t^* = I^{(1)}\big(e^{\delta t} J'(X_t^*)\big), \qquad t \geq 0. \tag{104}$$

5.2 Forward utility approach

The optimal investment problem with consumption has rarely been studied so far in the forward utility framework. The first authors who considered this problem were Berrier and Tehranchi [5]. Their approach was to follow the same idea that informed Musiela & Zariphopoulou [43, 44] in their definition of forward utilities in an optimal investment setting, which was to mimic the martingale and supermartingale property of the value function in the classical framework. Hence, they defined a forward performance criterion in this setting as follows:

Definition 3. A forward performance process for investment and consumption is a pair of $\mathbb{F}$-adapted processes $U^X(\cdot,\cdot), U^C(\cdot,\cdot)$ such that

i) $x \mapsto U^X(x,\cdot),\ c \mapsto U^C(c,\cdot)$ are increasing and strictly concave,

ii) Let $\xi_t^{\pi,c} := \int_0^t U^C(c_s, s)ds + U^X(X_t^{\pi,c}, t), \quad t \geq 0$,
For each $\pi, c \in \mathcal{A}$, $\mathbb{E}[|\xi_t^{\pi,c}|] < \infty, \quad t \geq 0$ and $\xi^{\pi,c}$ is a supermartingale, i.e.

$$\mathbb{E}[\xi_t^{\pi,c}|\mathcal{F}_s] \leq \xi_s^{\pi,c}, \qquad 0 \leq s \leq t,$$

iii) ξ is a martingale for some $\pi^*, c^* \in \mathcal{A}$, i.e.

$$\mathbb{E}[\xi_t^*|\mathcal{F}_s] = \xi_s^*, \qquad 0 \leq s \leq t.$$

Remark 11. Note that for the choice $U^C = 0$ the above definition is equivalent to the one in the investment-only case (cf. Definition 1), which affirms the consistency of Definition 3. We also note that we could add initial conditions to the definition, which would be imposed on U^X since $\xi_0^{\pi,c}(\cdot) = U^X(\cdot, 0)$.

Berrier and Tehranchi give a dual characterisation for a forward utility pair, and under the additional assumption of U^X being monotone in the time argument (and thus of finite variation, which implies it is in the class of zero-volatility performance processes) they derive an SPDE satisfied by the forward performance pair. Under the assumption that

U^X has an Itô decomposition and is such that the Itô-Ventzell formula can be applied, Källblad [25] and El Karoui et al. [13] (whereby the latter provide the more rigorous treatment) independently derive an SPDE which must be satisfied by the pair (U^X, U^C). It links the drift of U^X to the volatility process, and also to the consumption performance process U^C through its convex conjugate $\tilde{U}^{(c)}$. The result is stated in following Theorem.

Theorem 5.1. Let (U^X, U^C) be a forward utility pair such that U^X has the Itô decomposition $dU^X(x,t) = b(x,t)dt + v(x,t) \cdot dW_t$, which is such that the Itô-Ventzell formula can be applied (cf. Chapter 3 in Kunita [30]). Then the pair (U^X, U^C) satisfies the SPDE

$$\begin{aligned} dU^X(x,t) =& \Big(\frac{1}{2} \frac{\|U_x^X(x,t)\lambda_t + \sigma_t^{tr}\sigma_t(\sigma_t^{tr}\sigma_t)^{-1}v_x(x,t)\|^2}{U_{xx}^X(x,t)} - \tilde{U}^{(c)}\big(U_x(x,t),t\big)\Big)dt \\ &+ v(x,t) \cdot dW_t. \end{aligned} \tag{105}$$

Proof. A detailed proof can be found in [13], Proposition 3.3 b), and a heuristic derivation is given in [25]. Essentially, one needs to apply the Itô-Ventzell formula to the process $U^X(X_t^{\pi,c}, t)$ to get the dynamics of $(\xi_t^{\pi,c})_{t\geq 0}$ and impose the martingale condition to obtain a representation for the drift term of U^X. □

As was already noted by Källblad [25], for $U^C = 0$ the SPDE takes the form of the SPDE (14) satisfied by a forward performance process as derived by Musiela & Zariphopoulou [46] in the situation of just considering utility from wealth and no consumption. This shows that the definition of investment and consumption performance processes contains investment-only problems as a special case and thereby provides a more general framework which is consistent with previous developments. Källblad also noted that the SPDE (105) for a given initial condition and a given consumption performance process U^C has no unique solution due to the flexibility in the choice of the volatility process $v(\cdot,\cdot)$. Therefore, she imposes the additional constraint on U^X to be monotone in the time argument, which leads to $v(x,t) = 0$. For these kinds of zero-volatility processes she provides a characterisation in terms of (random) auxiliary functions. Zero-volatility forward performance processes that allow an Itô decomposition are also studied by Chong & Liang [10], who use results from infinite horizon BSDEs, coupled with a process defined through an ODE, to characterise CRRA-type forward utilities. However, they assume throughout their paper that the consumption utility process is static, i.e. $U^C(c,t) = U^C(c,0) = \frac{c^p}{p}$, which severely limits the flexibility of their forward performance functions. In addition to a rigorous derivation of the SPDE (105) El Karoui et al. [13] also provide conditions for the existence and uniqueness of a solution to (105) and show that forward utility pairs of power form are separable in time and wealth.

5.3 Itô-type forward utility pair and corresponding optimal strategies

We use the result from Theorem 5.1 to - analogously to the investment-only case - define the class of *Itô-type forward performance processes of investment and consumption.* By Theorem 5.1 this class encompasses all forward performance pairs for which U^X has an Itô decomposition, which will again allow us to formulate general results. This new definition will also be useful for future research in this area to postulate results in a very general form.

Definition 4. We call a pair of functions $U^X(\cdot,\cdot), U^C(\cdot,\cdot) : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ an Itô-type forward utility pair, if

a) (U^X, U^C) is a forward performance pair of investment and consumption, i.e. satisfies Definition 3,

b) $U^X \in \mathcal{C}^{3,0}(\mathbb{R} \times \mathbb{R}_+)$,

c) U^X has the Itô decomposition

$$dU^X(x,t) = b(x,t)dt + v(x,t) \cdot dW_t, \tag{106}$$

where $v(\cdot,\cdot) \in \mathcal{C}^{2,0}(\mathbb{R}^m)$ and is $\mathbb{F}$-adapted, and $b(\cdot,\cdot)$ is given by

$$b(x,t) = \frac{1}{2} \frac{\|U_x^X(x,t)\lambda_t + \sigma_t^{tr}\sigma_t(\sigma_t^{tr}\sigma_t)^{-1}v_x(x,t)\|^2}{U_{xx}^X(x,t)} - \tilde{U}^{(c)}\big(U_x(x,t),t\big), \tag{107}$$

d) U^X and its local characteristics satisfy the conditions outlined in Appendix A.1, so that the Itô-Ventzell formula can be applied to the process $U^X(X_t^{\pi,c},t)$, $t \geq 0$, for any admissible wealth process $X^{\pi,c}$.

Theorem 5.2. Let (U^X, U^C) be an Itô-type forward utility pair as in Definition 4. Then the optimal controls are given by

$$\pi_t^* = (\sigma_t\sigma_t^{tr})^{-1}\sigma_t\Big(R^X(X_t^*,t)\lambda_t - \frac{1}{U_{xx}^X(X_t^*,t)}v_x(X_t^*,t)\Big), \qquad t \geq 0, \tag{108}$$

$$c_t^* = I^C\big(U_x^X(X_t^*,t)\big), \qquad t \geq 0, \tag{109}$$

where $R^X(\cdot,\cdot)$ is the risk tolerance function of U^X as defined in (12).

Proof. The result appears as part of the proof of Theorem 5.1 by applying first order conditions to the drift process of $\xi_t^{\pi,c}$. It can be found in [13], cf. Equation 3.12 and the reasoning in the proof of Proposition 3.3 b) therein, and also appears in the heuristic derivation of [25], Equation (7). □

Remark 12. We observe that the optimal investment strategy for an Itô-type forward utility pair has the same structural form as in the case without consumption (cf. Equation (33)). Thus, for $U^C = 0$, which implies that $I^C = 0$, we recover the same result as in Section 4, Theorem 4.2. The optimal consumption policy has the same form as in the classical approach by depending on the sensitivity of the (wealth) forward performance process U^X with respect to the wealth argument and the marginal inverse of the consumption utility, but without the impatience factor, since it is implicitly incorporated in the time-dependent process U^C.

As in the previous section, we now aim to find an explicit example for such a forward utility pair and compare its optimal policies with their classical counterparts. Since a characterisation for a general class of forward utility pairs which allow for explicit solutions has not been established yet (see Remark 14), we derive an explicit example for a power-type forward utility pair of investment and consumption, motivated by the observation from El Karoui et al. (cf. Proposition 4.5 in [13]) as well as the class of performance criteria consisting of a differential and stochastic inputs (cf. Theorem 4.1).

Lemma 3 (Power forward utility pair of investment and consumption)**.** Let N, Z be the stochastic processes given by (15) and (16) respectively. Then a forward utility pair of investment and consumption is given by

$$U^X(x,t) = \frac{1}{p}(\frac{x}{N_t})^p e^{f(t)} Z_t, \qquad t \geq 0, \tag{110}$$

$$U^C(x,t) = \frac{1}{p}(\frac{c}{N_t})^p K e^{f(t)} Z_t, \qquad t \geq 0, \tag{111}$$

where $K \neq 0$ is a constant and $f(\cdot)$ is given by

$$f(t) = \int_0^t \Big(\frac{1}{2} q \|\lambda_s + \phi_s - p\delta_s\|^2 + (p-1)K^{1-q} + p\delta_s^{tr}\big(\lambda_s + \phi_s - \frac{1}{2}(p+1)\delta_s\big)\Big) ds, \quad t \geq 0.$$

Proof. The proof is given in Appendix A.3. □

Observe that for the forward performance $\xi_t^{\pi,c}(X_t^{\pi,c}) = \int_0^t U^C(c_s,s)ds + U^X(X_t^{\pi,c},t)$ it holds that $\xi_0^{\pi,c}(x) = \frac{x^p}{p}$, so a power utility initial condition is satisfied.

Corollary 5.2.1. The optimal strategies for the forward performance pair defined in Lemma 3 are given by

$$\pi_t^* = (\sigma_t \sigma_t^{tr})^{-1} \sigma_t \Big(\frac{X_t^*}{(1-p)}(\lambda_t + \phi_t - \delta_t) + \delta_t X_t^*\Big), \qquad t \geq 0, \tag{112}$$

$$c_t^* = X_t^* K^{1-q}, \qquad t \geq 0. \tag{113}$$

Proof. The statement follows immediately from Theorem 5.2 and the explicit expressions for the forward utility pair (110) and (111) from Lemma 3. The computations are given in Appendix A.3. □

Remark 13. We note that the optimal consumption path is given by a constant fraction of the wealth process. Since the investor can freely choose K, she can also achieve any desired fraction of wealth for her optimal consumption strategy. However, once the parameter K is fixed, it cannot be updated. In other words, the agent consumes the same fraction of wealth throughout the whole investment period. The choice of K is restricted by the admissibility criteria. We observe that the consumption path is admissible in the case that $K \in [0,1]$. Similar to the reasoning in the previous section with regard to the choice of the volatility process, a more detailed research would be needed to investigate how the parameter K is determined. Moreover, the question whether one can relax the assumption of K being a constant remains open and is left to future research. However, since the forward utility pair in Lemma 3 is only a specific example of a power-type forward performance process, it would be sensible to first derive a more general class of forward performance processes of investment and consumption, and then investigate this class in more detail. This brings us to our next remark.

Remark 14. Due to the scarce literature on forward utilities of investment and consumption, a general class which allows for explicit solutions - analogous to the class characterised in Theorem 4.1 for the pure investment case - has not been established yet. First attempts for characterisations of specific classes of forward performance criteria of investment and consumption can be found in Section 4.2 of Källblad [25], who gives a characterisation of zero-volatility forward utility pairs. The characterisation involves auxiliary (random) functions, which makes the comparison and interpretation of the resulting optimal strategies cumbersome. We propose that, based on the observation that the choice $U^C = 0$ gives a performance process U^X that should satisfy Definition 1 in Section 3, a natural approach to derive a more general class would be to consider a member the class from the investment only case characterised by Theorem 4.1 for the process U^X, and then try to derive conditions on the consumption performance process U^C such that the pair satisfies Definition 3. To limit the scope of this thesis, this is left for future research.

5.4 Example and comparison of classical versus forward optimal strategy

We take a standard Black-Scholes model ($d = m = 1$) with constant model parameters λ, σ and investigate the optimal strategies with power-type utility processes, i.e. we take $U^{(1)}(a) = U^{(2)}(a) = \frac{a^p}{p}, p < 1, p \neq 0$, and for the forward utility pair we take the pair defined in Lemma 3. We get that the optimal strategies for the classical and the forward framework respectively are given by

- Classical framework finite horizon (Karatzas et al. [27], Example 7.5[13]),

$$\pi_t^* = \frac{\lambda}{\sigma}\frac{X_t^*}{(1-p)}, \qquad 0 \leq t \leq T, \tag{114}$$

$$c_t^* = X_t^*\Big(\frac{p-1}{\left(\frac{1}{2}q\lambda^2+\delta\right)}\left(e^{-(1-q)(\frac{1}{2}q\lambda^2+\delta)(T-t)}-1\right)\Big)^{-1}, \qquad 0 \leq t \leq T. \tag{115}$$

- Classical framework infinite horizon (Merton [33] equations (42),(43)):

$$\pi_t^* = \frac{\lambda}{\sigma}\frac{X_t^*}{(1-p)}, \qquad t \geq 0, \tag{116}$$

$$c_t^* = X_t^*\frac{\left(\frac{1}{2}q\lambda^2+\delta\right)}{(1-p)}, \qquad t \geq 0. \tag{117}$$

- Forward utility framework:

$$\pi_t^* = \frac{X_t^*}{(1-p)}\frac{(\lambda+\phi_t-\delta_t)}{\sigma} + \frac{\delta_t}{\sigma}X_t^*, \qquad t \geq 0, \tag{118}$$

$$c_t^* = X_t^* K^{1-q}, \qquad t \geq 0. \tag{119}$$

Remark 15. We observe that with regard to the optimal investment strategy, the optimal fraction of wealth invested in the risky asset is the same as in the case with no consumption (cf. Example 1 in Section 4). Hence, as in the previous section, we can replicate the classical framework by choosing $Z = N = 1$, so a zero-volatility forward utility pair.

[13] Note that since we have no utility from terminal wealth, the extra term $+e^{k(T-t)}$ of the function $p(t)$ for $k \neq 0$ in the formula of the optimal consumption path in the reference [27] disappears in our case, in order that our terminal condition $V(T,x) = 0$ is satisfied.

Further, we remark that for the choice[14]

$$K = \Big(\frac{\frac{1}{2}q\lambda^2 + \delta}{1-p}\Big)^{1-p} \tag{120}$$

we get the same optimal consumption policy as in the classical framework (in the infinite horizon case). In the finite horizon case though, due to its horizon-dependence, the classical optimal consumption path cannot be replicated with the forward utility approach.

6 Indifference pricing with classical versus forward utility

In this section we consider a scenario where the agent is faced with a random payoff at some fixed future time T and aims to find the optimal investment strategy for this period (for simplicity we assume no consumption). An example of such a random endowment would be a long or a short position in a European option with a given maturity. Thus, for these kinds of problems there is a natural time horizon inherent to the problem. As was noted by Hugonnier and Kramkov [20], in the case of complete markets, when perfect replication of the random endowment is possible, the problem is equivalent to a standard investment problem, but with adjusted initial capital. Hence, we choose to study an incomplete market, so we take our general market model introduced in Section 2 and assume $m > d$. Following the modelling approach suggested by Karatzas et al. [28] (cf. Section 7 therein), we introduce $m-d$ *fictitious* assets, given by the vector $Y = (Y^{(1)}, Y^{(2)}, ..., Y^{(m-d)})$ with dynamics given by

$$dY_t = \mathrm{diag}_{m-d}(Y_t)a_t(b_t dt + dW_t), \qquad Y_0 = y, \tag{121}$$

where $(a_t)_{t\geq 0} \in \mathbb{R}^{(m-d)\times m}$, $(b_t)_{t\geq 0} \in \mathbb{R}^m$ are $\mathbb{F}$-adapted processes which moreover satisfy $\int_0^T \|b_t\| + \|a_t\|^2 dt < \infty$, for all $T > 0$. W is the same m-dimensional Brownian motion as in (9). The augmented market (S, Y) then forms a complete market, however the assets Y are non-traded and thus cannot be used for hedging.

Since the market of traded assets is incomplete, the class of EMMs and the corresponding class of density processes are not a singleton, but are given by

$$\mathcal{M} := \{\mathbb{Q} : \frac{d\mathbb{Q}}{d\mathbb{P}}|_t = \mathcal{E}(-(\lambda+\nu)\cdot W)_t, \text{s.t. } \nu_t \in \mathcal{F}_t, \int_0^t \|v_s\|^2 ds < \infty, \sigma_t \nu_t = 0, t \geq 0\}$$
$$\mathcal{Z} := \{\frac{d\mathbb{Q}}{d\mathbb{P}} : \mathbb{Q} \in \mathbb{M}\}. \tag{122}$$

As a consequence, there is no unique price for the claim, since we would first need to pick a measure $\mathbb{Q} \in \mathcal{M}$ in order to be able to apply standard pricing techniques. One way to overcome this issue was introduced by Hodges and Neuberger [19], and this concept called *indifference pricing* was subsequently extended by various authors; for a good overview we refer the reader to Chapter 2 in the book by Carmona [8]. We will now quickly introduce the general idea of indifference pricing. The first step (after fixing the model, of course) is to fix a utility function, where typically one uses an exponential type

[14] Where in this case δ is the impatience factor from the classical approach and not the volatility process of Z, which is given by $\delta_t = 0$.

utility. Secondly, the objective is defined as maximising terminal utility of wealth and the random endowment, and the corresponding value function or performance process is computed. Then, one defines the utility indifference price as the price per claim such that the agent is indifferent between paying/receiving the price to enter the position with the random endowment or not entering the position. This gives an equation which implicitly defines the indifference price and can - in some models - be solved explicitly.

Herein, we consider the random endowment to be a European-type claim $h(S_T, Y_T)$ with maturity T, for a payoff function $h(\cdot,\cdot)$, which is such that the exponential moment condition (here we follow among others Monoyios [38] and Becherer [3])

$$\mathbb{E}[\exp\big((\alpha+\epsilon)nh(S_T,Y_T)\big)] < \infty, \qquad \text{for some } \epsilon > 0, \tag{123}$$

is satisfied for all fixed $n \in \mathbb{Z}$. An example for such a claim would be a volatility derivative in a stochastic factor model (then h is only a function of Y); or one could also think of a case where the assets Y might be traded, but perfect hedging is impracticable and one wants to restrict trading only to a smaller amount of assets. Such a situation could arise, for example, when h is a basket option on a large amount of assets.

A particularly useful tool to tackle the problem of optimal investment including a position of n units ($n > 0$ is a long and $n < 0$ a short position) in the European claim maturing at T is given by duality theory. The dual representation for the classical case has been established by Delbaen et al. [12], a nice treatment of duality methods in incomplete markets is given by Karatzas et al. [28]. The forward case has been studied in Leung et al. [31], but only in a 2-dimensional model with bounded coefficients. The main result obtained herein (Theorem 6.2) gives the indifference price of n claims in a European-type option in a general Itô-process market under a time-monotone, exponential forward performance criterion.

As before, we will first give a quick overview over the approach and main results from the traditional framework, and will then look at the forward framework and derive the results in this model.

6.1 Classical approach

As noted above, duality methods play an important role in the classical indifference pricing framework. Hence we start by introducing the primal and dual value function process, which are defined by

$$\text{Primal}: \qquad u^{(n)}(X_t^\pi, Y_t, t) = \operatorname*{ess\,sup}_{\pi\in\mathcal{A}} \mathbb{E}[U\big(X_T + nh(Y_T,S_T)\big)|\mathcal{F}_t] \tag{124}$$

$$\text{Dual}: \qquad v^{(n)}(s, Y_t, t) = \operatorname*{ess\,inf}_{Z\in\mathcal{Z}} \mathbb{E}[\tilde{U}(s\frac{Z_T}{Z_t}) + s\frac{Z_T}{Z_t}nh(Y_T,S_T)|\mathcal{F}_t]. \tag{125}$$

Definition 5. The indifference price $p^{(n)}(t,x,y)$ is defined as the solution to the equation

$$u^{(n)}(t, x - np^{(n)}(x,y,t), y) = u^{(0)}(t,x), \qquad 0 \le t \le T. \tag{126}$$

Intuitively speaking, the indifference price is defined as the price such that the agent is indifferent between selling/buying n units of the claim for this price or not entering this position at all. Classically, one chooses exponential preferences for the agent as one can factor out wealth, which implies that exponential preferences yield prices which are independent of wealth and thus allow for tractable solutions, as it turns out.

Following theorem gives the per-claim exponential utility indifference price for n units of the European claim $h(Y_T, S_T)$. It generalises Theorem 4.5 and Lemma 4.8 from Monoyios [38], who shows these results for $n = -1$.

Theorem 6.1. For the choice of an exponential utility function $U(x) = -\exp(-\alpha x)$ for some $\alpha > 0$, the primal value function and the indifference price per claim for a position of n units in the claim $h(Y_T, S_T)$ are given for $0 \le t \le T$ by

$$u^{(n)}(x,y,t) = -\exp\Big(-\alpha x - \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{M}}\{\mathcal{H}_t(\mathbb{Q}|\mathbb{P}) + n\alpha\mathbb{E}^{\mathbb{Q}}[h(Y_T,S_T)|\mathcal{F}_t]\Big), \tag{127}$$

$$p^{(n)}(x,y,t) = \begin{cases} \operatorname{ess\,inf}_{\mathbb{Q}\in\mathcal{M}}\{\mathbb{E}^{\mathbb{Q}}[h(Y_T,S_T)|\mathcal{F}_t] + \frac{1}{n\alpha}\mathcal{H}_t(\mathbb{Q}|\mathbb{Q}^E)\}, & n \ge 0, \\ \operatorname{ess\,sup}_{\mathbb{Q}\in\mathcal{M}}\{\mathbb{E}^{\mathbb{Q}}[h(Y_T,S_T)\mathcal{F}_t] - \frac{1}{|n|\alpha}\mathcal{H}_t(\mathbb{Q}|\mathbb{Q}^E)\}, & n \le 0, \end{cases} \tag{128}$$

where $\mathcal{H}_t(\mathbb{Q}|\mathbb{P}) := \mathbb{E}^{\mathbb{Q}}[\log\big(\frac{Z_T^{\mathbb{Q}}}{Z_t\mathbb{Q}}\big)|\mathcal{F}_t]$ is the *conditional relative entropy* between $\mathbb{Q}$ and $\mathbb{P}$ and $\mathbb{Q}^E$ denotes the minimiser over all martingale measure $\mathbb{Q} \in \mathcal{M}$ of $\mathcal{H}_t(\mathbb{Q}|\mathbb{P})$, and is called the *minimal entropy measure*[15].

Proof. The proof is given in Appendix A.4. A slightly different proof for a similar statement is given in Monoyios [38], who considers the case $n = -1$ in his Theorem 4.5. □

Corollary 6.1.1. The so-called *marginal utility-based price* (MUBP) is given for the limit $n \to 0$ and represents the price for an infinitesimal small position in the claim. We observe that

$$\hat{p}_t = \lim_{n\to 0} p^{(n)}(x,y,t) = \mathbb{E}^{\mathbb{Q}^E}[h(Y_T)|\mathcal{F}_t], \qquad 0 \le t \le T, \tag{129}$$

so the MUBP for the classical framework is given by the expectation of the payoff under the minimal entropy measure. This is a well-known result, see for example Monoyios [36] Theorem 5.

6.2 Forward utility approach

In the forward utility framework, the performance process is given by

$$V^{(n)}(X_t^\pi, Y_t, t) = \operatorname*{ess\,sup}_{\pi\in\mathcal{A}} \mathbb{E}[U_T(X_T^\pi + nh(Y_T,S_T))|\mathcal{F}_t], \tag{130}$$

for some forward performance criterion $(U_t(\cdot))_{t\ge 0}$. We will focus on time-monotone criteria in order to get explicit solutions. The use of non-zero volatility forward performance processes in this problem has not been studied yet and will be left for future research. Again, we impose the initial condition of an exponential utility function so that we can compare the results with the classical framework. The utility process is thus given by $U_t(x) = -\exp\big(-\alpha x - \frac{1}{2}A_t\big)$, $t \ge 0$ (cf. Lemma 1). Hence the performance criterion reads

$$V^{(n)}(x,y,t) = \operatorname*{ess\,sup}_{\pi\in\mathcal{A}} \mathbb{E}[-e^{-\alpha\left(X_T^\pi + nh(Y_T,S_T)\right) + \frac{1}{2}\int_0^T \lambda_s^2 ds}|X_t = x, Y_t = y]. \tag{131}$$

[15] Proposition 4.1 of Kabanov and Stricker [22] asserts that the minimiser of the relative entropy (i.e. for $t = 0$) also minimises the conditional relative entropy, so we can simply call $\mathbb{Q}^E$ the minimal entropy measure.

Theorem 6.2. Under an exponential time-monotone forward performance criterion, the value function and the per-claim indifference price for a position of n units in the claim $h(Y_T, S_T)$ are given by

$$V^{(n)}(x,y,t) = -e^{-\frac{1}{2}A_t - \alpha x} e^{-\xi_t}, \qquad 0 \le t \le T, \quad (132)$$

$$\text{for} \quad \xi_t = \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{M}} \{\alpha n \mathbb{E}^{\mathbb{Q}}[h(Y_T,S_T)|\mathcal{F}_t] + \frac{1}{2}\mathbb{E}^{\mathbb{Q}}[\int_t^T \|\nu_s\|^2 ds|\mathcal{F}_t]\} \quad (133)$$

$$p_F^{(n)}(x,y,t) = \begin{cases} \operatorname{ess\,inf}_{\mathbb{Q}\in\mathcal{M}}\{\mathbb{E}^{\mathbb{Q}}[h(Y_T,S_T)|\mathcal{F}_t] + \frac{1}{n\alpha}\mathcal{H}_t(\mathbb{Q}|\mathbb{Q}^M)\}, & n \ge 0, \\ \operatorname{ess\,sup}_{\mathbb{Q}\in\mathcal{M}}\{\mathbb{E}^{\mathbb{Q}}[h(Y_T,S_T)|\mathcal{F}_t] - \frac{1}{|n|\alpha}\mathcal{H}_t(\mathbb{Q}|\mathbb{Q}^M)\}, & n \le 0. \end{cases} \quad (134)$$

Proof. We use the same idea as in the proof of Proposition 7 in Musiela & Zariphopoulou [46], which is to rewrite the payoff function in a way that we get an objective function which is of the form of the classical objective from the previous subsection (cf. Equation (124)). Then we can use the previous results and compute the value function and the corresponding indifference price.

Let $g(Y_T, S_T) := h(Y_T, S_T) - \frac{1}{2\alpha n}\int_0^T \lambda_s^2 ds$, then we can write

$$V^{(n)}(x,y,t) = \operatorname*{ess\,sup}_{\pi\in\mathcal{A}} \mathbb{E}[-e^{-\alpha(X_T^\pi + ng(Y_T,S_T))}|X_t = x, Y_t = y] \quad (135)$$

$$= \operatorname*{ess\,sup}_{\pi\in\mathcal{A}} \mathbb{E}[U\big(X_T + ng(Y_T,S_T)\big)|X_t = x, Y_t = y], \quad (136)$$

where U is the exponential utility function with parameter α. We observe that the value function now has the same form as in the classical framework (cf. Equation (124)), and also the dynamics of the underlying processes remain unchanged, thus we can use the results from Theorem 6.1 and get that the primal value function is given by

$$V^{(n)}(x,y,t) = -\exp\Big(-\alpha x - \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{M}}\{\mathcal{H}_t(\mathbb{Q}|\mathbb{P}) + \alpha n \mathbb{E}^{\mathbb{Q}}[g(Y_T)|\mathcal{F}_t]\}\Big). \quad (137)$$

Now we recall the definition of the density $Z^{\mathbb{Q}}$ and apply the Girsanov theorem to deduce

$$\mathcal{H}_t(\mathbb{Q}|\mathbb{P}) = \mathbb{E}^{\mathbb{Q}}[\log\big(\frac{Z_T^{\mathbb{Q}}}{Z_t^{\mathbb{Q}}}\big)|\mathcal{F}_t] \quad (138)$$

$$= \mathbb{E}^{\mathbb{Q}}[-\int_t^T (\lambda_s + \nu_s)dW_s - \frac{1}{2}\int_t^T \|\lambda_s + \nu_s\|^2 ds|F_t] \quad (139)$$

$$= \mathbb{E}^{\mathbb{Q}}[-\int_t^T (\lambda_s + \nu_s)dW_s^{\mathbb{Q}} + \frac{1}{2}\int_t^T \|\lambda_s + \nu_s\|^2 ds|F_t]. \quad (140)$$

We observe that $\lambda_t^{tr}\nu_t = \mu_t^{tr}\big((\sigma_t\sigma_t^{tr})^{-1}\big)^{tr}\sigma_t\nu_t = 0, \quad 0 \le t \le T$, by the definition of λ and the condition on ν_t (cf. (122)). Thus $\|\lambda_t + \nu_t\|^2 = \|\lambda_t\|^2 + \|\nu_t\|^2, \quad t \ge 0$. This implies that (137) reduces to

$$V^{(n)}(x,y,t) = -e^{\Big(-\frac{1}{2}A_t - \alpha x - \operatorname{ess\,inf}_{\mathbb{Q}\in\mathcal{M}}\{\alpha n\mathbb{E}^{\mathbb{Q}}[h(Y_T,S_T)|\mathcal{F}_t] + \frac{1}{2}\mathbb{E}^{\mathbb{Q}}[\int_t^T \|\nu_s\|^2 ds|\mathcal{F}_t]\}\Big)}. \quad (141)$$

To establish the second part of the theorem, we observe that for $n = 0$ the infimum is achieved by the choice $\nu_t = 0$, so that $u^{(0)}(x,t) = -e^{-\alpha x - \frac{1}{2}A_t}$. Thus we get that the indifference price is given by

$$p_F^{(n)}(x,y,t) = \begin{cases} \operatorname{ess\,inf}_{\mathbb{Q}\in\mathcal{M}}\{\mathbb{E}^{\mathbb{Q}}[h(Y_T)|\mathcal{F}_t] + \frac{1}{n\alpha}\frac{1}{2}\mathbb{E}^{\mathbb{Q}}[\int_t^T \|\nu_s\|^2 ds|\mathcal{F}_t]\}, & n \geq 0, \\ \operatorname{ess\,sup}_{\mathbb{Q}\in\mathcal{M}}\{\mathbb{E}^{\mathbb{Q}}[h(Y_T)|\mathcal{F}_t] - \frac{1}{|n|\alpha}\frac{1}{2}\mathbb{E}^{\mathbb{Q}}[\int_t^T \|\nu_s\|^2 ds|\mathcal{F}_t]\}, & n \leq 0. \end{cases} \tag{142}$$

We use Lemma 4.1 in Monoyios [38], which states that for $\mathbb{Q}_1, \mathbb{Q}_2 \in \mathcal{M}$

$$\frac{d\mathbb{Q}_1}{d\mathbb{Q}_2}|_t = \frac{\frac{d\mathbb{Q}_1}{d\mathbb{P}}|_t}{\frac{d\mathbb{Q}_2}{d\mathbb{P}}|_t}, \qquad t \geq 0, \tag{143}$$

to compute that $\frac{d\mathbb{Q}}{d\mathbb{Q}^M}|_t = \exp\big(-\int_t^T \nu_s dW_s^{\mathbb{Q}} + \frac{1}{2}\int_t^T \|\nu_s\|^2 ds\big)$, which in turn shows that $\mathcal{H}_t(\mathbb{Q}|\mathbb{Q}^M) = \frac{1}{2}\mathbb{E}^{\mathbb{Q}}[\int_t^T \|\nu_s\|^2 ds|\mathcal{F}_t]$. Hence (142) reduces to

$$p_F^{(n)}(x,y,t) = \begin{cases} \operatorname{ess\,inf}_{\mathbb{Q}\in\mathcal{M}}\{\mathbb{E}^{\mathbb{Q}}[h(Y_T)|\mathcal{F}_t] + \frac{1}{n\alpha}\mathcal{H}_t(\mathbb{Q}|\mathbb{Q}^M)\}, & n \geq 0, \\ \operatorname{ess\,sup}_{\mathbb{Q}\in\mathcal{M}}\{\mathbb{E}^{\mathbb{Q}}[h(Y_T)|\mathcal{F}_t] - \frac{1}{|n|\alpha}\mathcal{H}_t(\mathbb{Q}|\mathbb{Q}^M)\}, & n \leq 0. \end{cases} \tag{144}$$

□

Corollary 6.2.1. The MUBP in the forward framework under a time-monotone exponential performance criterion is given by

$$\hat{p}_{F;t} = \lim_{n\to 0} p^{(n)}(x,y,t) = \mathbb{E}^{\mathbb{Q}^M}[h(Y_T)|\mathcal{F}_t], \qquad 0 \leq t \leq T. \tag{145}$$

Remark 16. We note following striking difference between the MUBP in the classical and the forward utility framework. While in the classical framework the MUBP is given by the expectation of the payoff with respect to the minimal entropy measure (which depends on the model choice), the MUBP in the forward utility framework is always the expectation of the payoff with respect to the minimum martingale measure, regardless of the underlying model. Also for the indifference pricing formula, we get the same representation of the indifference price for both approaches (compare (241) and (144)), but while the classical framework takes the relative entropy with respect to the minimal entropy measure, the forward framework uses the minimal martingale measure instead.

Remark 17. If λ_t is deterministic, then we have that the minimal entropy measure is the minimiser of

$$\frac{1}{2}\int_t^T \|\lambda_s\|^2 ds + E^{\mathbb{Q}}[\int_t^T \|\nu_s\|^2 ds|\mathcal{F}_t] \to \inf_{\mathbb{Q}\in\mathcal{M}}!, \tag{146}$$

which is minimised by choosing $\nu_s = 0$. So in this case, $\mathbb{Q}^E = \mathbb{Q}^M$ and the classical and forward indifference price coincide.

Remark 18. Proposition 7 in Musiela & Zariphopoulou [31] shows a similar result to Theorem 6.2, but is given only for a long position in an American option in a stochastic volatility market model, i.e. $d = 1, m = 2$, with bounded coefficients for the stock price dynamics. We note that our theorem could easily be extended to American options in a similar manner than was done in [31] by writing it as a combined optimal control and optimal stopping problem. However, since we herein consider the problem with a natural time horizon, we decide to avoid the technical details and only consider European-type options.

6.3 Example and comparison of classical versus forward indifference price

A simple example is the so called *basis risk model*, where we have one stock and one non-traded factor ($m = 2, d = 1$) and random coefficients $\lambda_t, \sigma_t, a_t, b_t$. We consider a claim $h(\cdot)$ written only on the non-traded asset Y.

Lemma 4. In the above-described basis risk model with random coefficients, the per-unit indifference price $p^{(n)}(x, y, t)$ and $p_F^{(n)}(x, y, t)$ for n units of the claim $h(Y_T)$, in the classical and in the forward framework respectively, is given by

$$p^{(n)}(x, y, t) = -\frac{\gamma}{n\alpha} \log\Big(\mathbb{E}^{\mathbb{Q}^E}[e^{-\alpha\frac{n}{\gamma}h(Y_t)}|\mathcal{F}_t]\Big), \qquad \gamma = \frac{1}{1-\rho^2}, \tag{147}$$

$$p_F^{(n)}(x, y, t) = -\frac{\gamma}{n\alpha} \log\Big(\mathbb{E}^{\mathbb{Q}^M}[e^{-\alpha\frac{n}{\gamma}h(Y_t)}|\mathcal{F}_t]\Big), \qquad 0 \le t \le T, \tag{148}$$

Proof. In the classical framework, Proposition 5 in Monoyios [35] shows[16] that the value function in the traditional framework is given by

$$u^{(n)}(x, y, t) = -e^{-\alpha x}\Big(f(t, y)\Big)^{\gamma}, \qquad \gamma = \frac{1}{1-\rho^2}, \tag{149}$$

$$f(t, y) = \mathbb{E}^{\mathbb{Q}^M}[e^{-\alpha\frac{n}{\gamma}h(Y_t)-\frac{1}{2\gamma}\int_t^T \lambda_s^2 ds}|\mathcal{F}_t], \qquad 0 \le t \le T, \tag{150}$$

where ρ is the correlation of the 2 driving Brownian motions. Using this in the indifference price formula (126) we get that

$$p^{(n)}(t, x, y) = -\frac{\gamma}{n\alpha} \log\Big(\frac{\mathbb{E}^{\mathbb{Q}^M}[e^{-\alpha\frac{n}{\gamma}h(Y_T)-\frac{1}{2\gamma}\int_t^T \lambda_s^2 ds}|\mathcal{F}_t}{\mathbb{E}^{\mathbb{Q}^M}[e^{-\frac{1}{2\gamma}\int_t^T \lambda_s^2 ds}|\mathcal{F}_t]}\Big). \tag{151}$$

Corollary 3 in Monoyios [35] shows that

$$Z_T^{\mathbb{Q}^{\mathbb{E}},\mathbb{Q}^{\mathbb{M}}} = \frac{d\mathbb{Q}^{\mathbb{E}}}{d\mathbb{Q}^{\mathbb{M}}}\Big|_{\mathcal{F}_T} = \frac{\exp\big(-\frac{1}{2}(1-\rho^2)\int_0^T \lambda_s^2 ds\big)}{\mathbb{E}^{\mathbb{Q}^{\mathbb{M}}}[\exp\big(-\frac{1}{2}(1-\rho^2)\int_0^T \lambda_s^2 ds\big)|\mathcal{F}_T]}. \tag{152}$$

By Bayes' theorem for conditional expectations (Shreve [55], Lemma 5.2.2) we have that for an $\mathcal{F}_T$-measurable random variable C it holds that

$$\mathbb{E}^{\mathbb{Q}^E}[C|\mathcal{F}_t] = \frac{1}{Z_t^{\mathbb{Q}^{\mathbb{E}},\mathbb{Q}^{\mathbb{M}}}}\mathbb{E}^{\mathbb{Q}^M}[C Z_T^{\mathbb{Q}^{\mathbb{E}},\mathbb{Q}^{\mathbb{M}}}|\mathcal{F}_t], \qquad 0 \le t \le T. \tag{153}$$

Hence we identify the density in the formula above (151) and apply Bayes' theorem to obtain

$$p^{(n)}(t, x, y) = -\frac{\gamma}{n\alpha} \log\Big(\mathbb{E}^{\mathbb{Q}^E}[e^{-\alpha\frac{n}{\gamma}h(Y_T)}|\mathcal{F}_t]\Big), \qquad 0 \le t \le T. \tag{154}$$

For the forward utility framework, we use the same trick as in the proof of Theorem 6.2 and 'absorb' the time-dependent term of the exponential time-monotone utility by

[16] In [37] they only consider $n = -1$. However, their requirement on the claim payoff $B \in \mathcal{F}_T$ is that it is such that all expectations are well-defined. Thus, if we set $-B(\cdot) = nh(\cdot)$, then the result of Proposition 5 still holds true since by assumption $h(\cdot)$ satisfies the exponential moment conditions.

defining a new payoff function $g(Y_T) = h(Y_T) - \frac{1}{n\alpha}\frac{1}{2}\int_t^T \lambda_s^2 ds$, so that we recover the same form as in the objective function of the classical approach. The value function and the indifference price can then be computed to take the form

$$V^{(n)}(t,x,y) = -\, e^{-\alpha x + \frac{1}{2}A_t}\mathbb{E}^{\mathbb{Q}^M}[e^{-\alpha\frac{n}{\gamma}h(Y_t)}|\mathcal{F}_t], \qquad 0 \le t \le T, \tag{155}$$

$$p_F^{(n)}(t,x,y) = -\,\frac{\gamma}{n\alpha}\log\left(\mathbb{E}^{\mathbb{Q}^M}[e^{-\alpha\frac{n}{\gamma}h(Y_t)}|\mathcal{F}_t]\right), \qquad 0 \le t \le T. \tag{156}$$

□

Remark 19. We observe that for $(\lambda_t)_{0\le t\le T}$, such that $\int_t^T \lambda_s^2 ds$ is independent of Y_T for all $0 \le t \le T$, the indifference price for the two approaches coincide, so in particular for deterministic λ (in that case also the marginal utility-based prices are the same). In general though, the two prices are not necessarily the same.

7 Conclusion

In summary, we have reviewed, summarised and unified the existing literature on forward performance processes. We have generalised existing results on optimal investment policies by introducing a new and very general class of forward performance processes called *Itô-type forward utilities*, which encompasses the most widely used forward utility functions in the literature, as we have shown. We find that the optimal investment strategy under an Itô-type forward performance criterion is given by the myopic portfolio plus a correction term, which depends on the derivative of the volatility process of the forward performance process with respect to the wealth argument. This leads us to conclude that, in a general incomplete Itô-process model, Itô-type forward utilities whose volatility process is not wealth-dependent always yield the myopic portfolio, which generalises a similar observation which has previously been established only for time-monotone (i.e. zero volatility) forward utilities in a two factor model. Moreover, we find that optimal investment strategies for Itô-type forward utilities ignore unhedgeable risk factors, as long as the external stochastic factor is not explicitly introduced into the volatility process of the forward utility. The consideration of specific market models and an explicit computation of the strategies therein emphasises these findings and puts the optimal strategies of the new approach into perspective by comparing them with their classical counterparts. We find that, except for the cases when the classical strategy depends on the time horizon, the classical strategies can be replicated by a certain choice of the volatility parameter. We also show how earlier results for the most widely used utility functions, consisting of a differential and stochastic inputs, follow as a special case of our general optimal policy for Itô-type forward utilities in an incomplete Itô-process market.

When adding consumption to the problem, we were able to extend the definition of Itô-type forward performance processes - using characterisation results from El Karoui et al. [13] and Källblad [25] - in a way which is consistent with the previous definition. We observe that the structure of the optimal investment strategy under Itô-type forward utilities stays the same as in the pure investment problem, and the optimal consumption path takes a similar form than in the classical approach using the marginal inverse of the utility of consumption process. The literature on forward performance processes for investment and consumption problems is scarce, and apart from a zero-volatility class represented by auxiliary functions, which has been introduced by Källblad [25], no class whose members have an explicit representation has been established so far. Thus, we derive an explicit power-type member of Itô-type forward utilities and compute the resulting optimal policies in a standard Black-Scholes model. We compare with the optimal policies from the classical approach and find that a scaling parameter allows us to replicate the classical infinite horizon *Merton strategy*. The choice of the scaling parameter is rather arbitrary though, and allows the agent to achieve any fraction of wealth as optimal consumption path (with this fraction staying constant over the whole investment period), which leaves the question how this parameter can be calibrated to an individual's preferences an open one, and is left to future research.

Lastly, we establish a formula for the indifference price for a European type claim under a time-monotone exponential performance criterion in a general incomplete Itô-type market. We consider a general claim - on the traded and/or non-traded factors - which satisfies an exponential moment condition. We observe that the formulae for the classical and forward exponential utility indifference price are essentially the same in terms of structure, with the striking difference that the classical formula has a penalisation term which includes the relative entropy with respect to the minimal entropy measure, while the forward formula penalises using the relative entropy with respect to the minimal martingale measure. As a consequence, we conclude that the indifference pricing formulae are the same if the market price of risk process is deterministic. We further conclude that the marginal utility-based price is the expectation of the payoff of the claim with respect to the minimal martingale measure (in the forward framework) as opposed to with respect to the minimal entropy measure (in the classical framework). This generalises similar results that have previously appeared in the literature for specific model and/or claim choices.

To conclude, we highlight some opportunities for future research and suggest approaches to look at these problems. Most notably, we find that while there is extensive literature on the dual side of classical investment problems (see, e.g., the bibliographic remarks of Chapter 7 in the book by Pham [50]), it has rarely been considered for the forward framework (to the best of our knowledge, only Zitkovic [59], Berrier et al. [5] and to some extent Källblad et al. [26] have considered dual characterisations for forward utilities). Hence, it could be informative to explore the dual side of forward performance processes. Since the minimal entropy measure often plays an important role in the characterisations of optimality on the dual side for the classical approach, the observations made in Section 6 for the exponential utility indifference pricing formulae in combination with the observation from Section 4 that unhedgeable risk factors are ignored lead us to conjecture that similar formulae that have been proven for the classical case can also be established in the forward case, but with the minimal entropy measure being replaced by the minimal martingale measure (at least for time-monotone forward utilities).

Secondly, we noted that the (derivative of the) volatility process of the forward utility appears explicitly in the optimal strategy. In particular, for the popular class of forward performance criteria constructed from a differential and stochastic inputs, the volatility process of the *market view process* prescribes the optimal strategy. In spite of that, there has so far been no investigation that tries to answer the question how this process is chosen (either explicitly or implicitly by her investment decisions) by the agent. An understanding of this procedure could provide valuable insights on the optimal investment policies and could allow for calibration and consequently actual implementation of forward performance investment strategies. A potential way to study this problem could be to use empirical data from an agent's past investment decisions and calibrate the market view process to these decisions. This approach would assume that the decisions were indeed optimal according to such a type of forward performance criterion.

Thirdly, the first basic definitions and characterisations of forward utilities for investment and consumption can be extended and studied in more detail. Our general definition of Itô-type forward performance processes can prove useful for obtaining general results. A first direction could be to characterise a class of forward utilities which allow for tractable and explicit representations - analogue to the popular class of forward utilities constructed from a differential and stochastic inputs in the pure investment case. This would allow for comparisons of the resulting optimal policies to the classical case in a similar fashion than we have done in Section 4 of this thesis, which could provide interesting insights.

Another direction for future research could be to consider variations of the forward utility functions. For example, one could add a jump process to the dynamics of the forward utility function, which could capture sudden shocks to an agent's preferences. Alternatively, in the same spirit as the recent approach taken by He et al. [17] who use rank-dependent utilities, one could think about varying the classical preferences and, e.g., consider behavioural preferences, which originate in the seminal papers by Kahnemann & Tversky [23, 24], who use insights of experimental psychology to establish a new theory on decision making under risk, called *prospect theory*. A framework for portfolio selection under preferences which are based on prospect theory has been established by Jin & Zhou [21]. Investigating the forward side of this framework could be an interesting endeavor.

As one can see, even though there is already extensive literature on the forward utility framework, there are still manifold opportunities to further investigate and extend this theory. The unification and in particular the general definitions in this thesis will be valuable as they provide a common language to formulate general results, and to keep any future extensions tractable within the existing framework.

Appendix

A.1 Itô-Ventzell formula

We outline briefly below the technical details for the application of the Itô-Ventzell formula in our setting. For a comprehensive treatment containing the general definitions and theorems we refer the reader to the book by Kunita [30].

Let $U(\cdot,\cdot)$ be a random field with Itô decomposition (cf. 13)

$$dU(x,t) = b(x,t)dt + v(x,t)\cdot dW_t. \tag{157}$$

The *local characteristics* of $U(x,t)$ are given by $\big(a(\cdot,\cdot,\cdot), b(\cdot,\cdot)\big)$, where $a(x,y,t)$ is the integrand in the join quadratic variation of $U(x,t)$ and $U(y,t)$, i.e.

$$\langle U(x,\cdot), U(y,\cdot)\rangle_t = \int_0^t a(x,y,s)ds, \tag{158}$$

which implies that $a(x,y,t) = v(x,t)^{tr}v(y,t)$ in our case. For a twice continuously differentiable (with respect to x and y) function $g(\cdot,\cdot,\cdot) : \mathbb{D}\times\mathbb{D}\times\mathbb{R}_+ \to \mathbb{R}$, where $\mathbb{D}$ denotes the domain of $x \mapsto v(x,t)$, we define the semi-norms

$$\|g(t)\|^{\sim}_{1:K} := \sup_{x,y\in K} \frac{|g(x,y,t)|}{(1+|x|)(1+|y|)} + \sup_{x,y\in K} |g_{xy}(x,y,t)| \tag{159}$$

$$\|g(t)\|^{\sim}_{1+0:K} := \|g(t)\|^{\sim}_{1:K} + \sup_{\substack{x,y,\hat{x},\hat{y}\in K\\ x\neq\hat{x}, y\neq\hat{y}}} |g_{xy}(x,y,t) - g_{xy}(\hat{x},y,t) - g_{xy}(x,\hat{y},t) + g_{xy}(\hat{x},\hat{y},t)|, \tag{160}$$

for some subset $K \subseteq \mathbb{D}$. Similarly, for a continuously differentiable function (with respect to the spatial argument) $f(\cdot,\cdot) : \mathbb{D}\times\mathbb{R}_+ \to \mathbb{R}$ we define the semi-norms

$$\|f(t)\|_{1:K} := \sup_{x\in K} \frac{|f(x,t)|}{(1+|x|)} + \sup_{x\in K} |f_x(x,t)| \tag{161}$$

$$\|f(t)\|_{1+0:K} := \|f(t)\|_{1:K} + \sup_{\substack{x,\hat{x}\in K\\ x\neq\hat{x}}} |f_x(x,t) - f_x(\hat{x},t)|, \tag{162}$$

for some $K \subseteq \mathbb{D}$.

Assumption $\triangle$:

The drift $b(\cdot,\cdot)$ and volatility process $v(\cdot,\cdot)$ of $U(\cdot,\cdot)$ are such that $b(\cdot,t), v(\cdot,t) \in \mathcal{C}^{1,0}$, for each $t \geq 0$, and the local characteristics $\big(a(\cdot,\cdot,\cdot), b(\cdot,\cdot)\big)$ of $U(\cdot,\cdot)$ satisfy

$$\int_0^T \|a(t)\|^{\sim}_{1+0:K}dt < \infty, \qquad T \geq 0, \tag{163}$$

$$\int_0^T \|b(t)\|_{1+0:K}dt < \infty, \qquad T \geq 0, \tag{164}$$

almost surely, for every compact $K \subset \mathbb{D}$.

Lemma 5. Let $U(\cdot,\cdot)$ as in (157) be $\mathbb{F}$-adapted, $U(\cdot,\cdot) \in C^{2,0}$ and such that assumption $\triangle$ holds. Then for every admissible wealth process X the Itô-Ventzell formula holds

$$dU(X_t,t) = b(X_t,t)dt + v(X_t,t)\cdot dW_t + U_x(X_t,t)dX_t + \frac{1}{2}U_{xx}(X_t,t)d\langle X\rangle_t + v_x(X_t,t)^{tr}d\langle X,W\rangle_t. \tag{165}$$

Proof. We refer to the proof of Theorem 3.3.1 in [30]. □

A.2 Proof of Lemma 1

Proof. We derive the forward performance processes characterised by Theorem 4.1, which consist of a differential and stochastic inputs. We impose a power, logarithmic and exponential utility function respectively as an initial condition, so that we can associate the forward utility process with a classical value function. We recall the characterisation theorem (Theorem 4.1), which says that a forward utility function is given by

$$U_t(x) = u\Big(\frac{x}{N_t}, A_t\Big)Z_t, \qquad t \geq 0 \tag{166}$$

$$U_0(x) = U(x), \tag{167}$$

where u satisfies $u_t = \frac{1}{2}\frac{u_x^2}{u_{xx}}$ (cf. (18)) and N, Z, A are given by (15)-(17). We first derive the differential input functions $u(\cdot,\cdot)$ corresponding to a power, logarithmic and exponential initial condition (21)-(23).

- **Case 1:** $U(x) = \frac{x^p}{p}$
  We propose the ansatz: $u(x,t) = \frac{x^p}{p}f(t) + g(t)$, with $f(0) = 1, g(0) = 0$.

$$u_x(x,t) = x^{p-1}f(t) \tag{168}$$

$$u_{xx}(x,t) = (p-1)x^{p-2}f(t) \tag{169}$$

$$u_t(x,t) = \frac{x^p}{p}f'(t) + g'(t). \tag{170}$$

  Now using the PDE (18) we get that for every x and t

$$\frac{x^p}{2(p-1)}f(t) = \frac{x^p}{p}f'(t) + g'(t), \tag{171}$$

  which implies that

$$f'(t) = \frac{1}{2}\frac{p}{p-1}f(t) \tag{172}$$

$$g'(t) = 0. \tag{173}$$

  We observe that $g(\cdot)$ is a constant and $f(\cdot)$ is of exponential form. Using the initial conditions and the notational convention $q = \frac{p}{p-1}$ we obtain

$$g(t) = 0 \qquad t \geq 0, \tag{174}$$

$$f(t) = \exp\Big(\frac{1}{2}qt\Big), \qquad t \geq 0. \tag{175}$$

  Hence we get that the differential input for the power forward utility function is given by

$$u(x,t) = \frac{x^p}{p}e^{\frac{1}{2}qt}, \qquad t \geq 0. \tag{176}$$

- **Case 2:** $U(x) = \log(x)$
  We propose the ansatz: $u(x,t) = \log(x)f(t) + g(t)$, with $f(0) = 1, g(0) = 0$

$$u_x(x,t) = \frac{1}{x}f(t) \tag{177}$$

$$u_{xx}(x,t) = -\frac{1}{x^2}f(t) \tag{178}$$

$$u_t(x,t) = \log(x)f'(t) + g'(t). \tag{179}$$

Again we use the PDE (18) and compute

$$-\frac{1}{2}f(t) = \log(x)f'(t) + g'(t), \qquad t \geq 0, \quad x \in \mathbb{R}, \tag{180}$$

so we must have that $f'(t) = 0$. By the initial condition we get that $f \equiv 1$, hence we deduce that $g'(t) = -\frac{1}{2}, \quad t \geq 0$. This implies that

$$f(t) = 1, \qquad t \geq 0, \tag{181}$$

$$g(t) = -\frac{t}{2}, \qquad t \geq 0. \tag{182}$$

Thus the differential input of the logarithmic forward performance process is given by

$$u(x,t) = \log(x) - \frac{t}{2}, \qquad t \geq 0. \tag{183}$$

- **Case 3:** $U(x) = -e^{-\alpha x}$
  We propose the ansatz: $u(x,t) = -e^{-\alpha x}f(t)$, with $f(0) = 1$. Then

$$u_x(x,t) = \alpha e^{-\alpha x}f(t) \tag{184}$$

$$u_{xx}(x,t) = -\alpha^2 e^{-\alpha x}f(t) \tag{185}$$

$$u_t(x,t) = -e^{-\alpha x}f'(t). \tag{186}$$

As before by the PDE (18) $\frac{1}{2}f(t) = f'(t), \quad t \geq 0$, so with the initial condition we get that

$$f(t) = e^{\frac{t}{2}}, \qquad t \geq 0, \tag{187}$$

which yields the exponential case differential input function

$$u(x,t) = -e^{-\alpha x + \frac{t}{2}}, \qquad t \geq 0. \tag{188}$$

The respective optimal forward performance process is then obtained by using Definition (166) and plugging in the stochastic inputs into the respective differential input function. This gives us

1. Power forward utility: $$U_t(x) = \frac{1}{p}\big(\frac{x}{N_t}\big)^p e^{\frac{q}{2}A_t}Z_t, \qquad t \geq 0, \tag{189}$$
2. Logarithmic forward utility: $$U_t(x) = \Big(\log\big(\frac{x}{N_t}\big) - \frac{A_t}{2}\Big)Z_t, \qquad t \geq 0, \tag{190}$$
3. Exponential forward utility: $$U_t(x) = -\exp\Big(-\alpha\big(\frac{x}{N_t}\big) + \frac{A_t}{2}\Big)Z_t, \qquad t \geq 0. \tag{191}$$

Recalling that $R(x,t) = -\frac{\frac{\partial}{\partial x}U_t(x)}{\frac{\partial^2}{\partial x^2}U_t(x)}$, we take derivatives in the above expressions and obtain that

1. Power forward utility: $$R(x,t) = \frac{x}{1-p}, \qquad t \geq 0, \tag{192}$$
2. Logarithmic forward utility: $$R(x,t) = x, \qquad t \geq 0, \tag{193}$$
3. Exponential forward utility: $$R(x,t) = \frac{N_t}{\alpha}, \qquad t \geq 0, \tag{194}$$

which concludes the proof. □

A.3 Proof of Lemma 3 (Power forward utility pair of investment and consumption) and Corollary 5.2.1

Proof of Lemma 3. We want to find a pair of functions U^X, U^C satisfying the definition of an Itô-type forward utility pair (Definition 4) with initial condition $U(x) = \frac{x^p}{p}$, i.e. U^X must satisfy

$$dU^X(x,t) = \Big(\frac{1}{2}\frac{\|U_x^X(x,t)\lambda_t + \sigma_t^{tr}\sigma_t(\sigma_t^{tr}\sigma_t)^{-1}v_x(x,t)\|^2}{U_{xx}^X(x,t)} - \tilde{U}^{(c)}\big(U_x(x,t),t\big)\Big)dt + v(x,t)\cdot dW_t. \tag{195}$$

Motivated by the observation from El Karoui et al. [13] that power-type forward utility pairs of investment and consumption have a separable solution, and also informed by the forward performance processes of investment characterised in Theorem 4.1 we propose the ansatz

$$U^X(x,t) = u^{(1)}(\frac{x}{N_t},t)Z_t, \qquad u^{(1)}(x,t) = \frac{x^p}{p}e^{f(t)}, \tag{196}$$

$$U^C(c,t) = u^{(2)}(\frac{c}{N_t},t)Z_t, \qquad u^{(2)}(c,t) = \frac{c^p}{p}Ke^{f(t)}, \quad K \neq 0, \tag{197}$$

where N, Z are the benchmark and market view process as defined in (15) and (16) respectively, and $f(\cdot)$ has initial condition $f(0) = 0$.

Step 1: We start by computing the dynamics of U^X

$$dU^X(x,t) = d\Big(u^{(1)}(\frac{x}{N_t},t)Z_t\Big) = Z_t d\Big(u^{(1)}(\frac{x}{N_t},t)\Big) + u^{(1)}(\frac{x}{N_t},t)dZ_t + d\left\langle u^{(1)}(\frac{x}{N},\cdot), Z\right\rangle_t. \tag{198}$$

We compute the dynamics of each term separately using Itô's formula

$$d\Big(u^{(1)}(\frac{x}{N_t},t)\Big) = u_t^{(1)}(\frac{x}{N_t},t)dt + u_x^{(1)}(\frac{x}{N_t},t)xd(\frac{1}{N_t}) + \frac{1}{2}u_{xx}^{(1)}(\frac{x}{N_t},t)x^2 d\left\langle\frac{1}{N}\right\rangle_t \tag{199}$$

$$= \Big(u_t^{(1)} + (\frac{x}{N_t}\delta_t^{tr}\delta_t - \frac{x}{N_t}\delta_t^{tr}\lambda_t)u_x^{(1)} + \frac{1}{2}\frac{x^2}{N_t^2}\delta_t^{tr}\delta_t u_{xx}^{(1)}\Big) - u_x^{(1)}\frac{x}{N_t}\delta_t \cdot dW_t, \tag{200}$$

where we dropped the arguments $(\frac{x}{N_t},t)$ in the second line for readability (and we will continue to do so). We recall that

$$dZ_t = Z_t\phi_t \cdot dW_t, \tag{201}$$

thus we get that

$$d\left\langle u^{(1)}(\frac{x}{N},\cdot), Z\right\rangle_t = -u_x^{(1)}\frac{x}{N_t}\delta_t^{tr}\phi_t Z_t dt. \tag{202}$$

Combining the terms and plugging them back into (198) we get

$$dU^X(x,t) = Z_t\Big(u_t^{(1)} + (\frac{x}{N_t}\delta_t^{tr}\delta_t - \frac{x}{N_t}\delta_t^{tr}\lambda_t)u_x^{(1)} + \frac{1}{2}\frac{x^2}{N_t^2}\delta_t^{tr}\delta_t u_{xx}^{(1)} - u_x^{(1)}\frac{x}{N_t}\delta_t^{tr}\phi_t\Big)dt + Z_t\Big(u^{(1)}\phi_t - u_x^{(1)}\frac{x}{N_t}\delta_t\Big)\cdot dW_t, \tag{203}$$

so the drift $b(\cdot,\cdot)$ and volatility process $v(\cdot,\cdot)$ of the Itô decomposition of $U^X(\cdot,\cdot)$ are given by

$$b(x,t) = Z_t\Big(u_t^{(1)} + (\frac{x}{N_t}\delta_t^{tr}\delta_t - \frac{x}{N_t}\delta_t^{tr}\lambda_t)u_x^{(1)} + \frac{1}{2}\frac{x^2}{N_t^2}\delta_t^{tr}\delta_t u_{xx}^{(1)} - u_x^{(1)}\frac{x}{N_t}\delta_t^{tr}\phi_t\Big) \tag{204}$$

$$v(x,t) = Z_t\Big(u^{(1)}\phi_t - u_x^{(1)}\frac{x}{N_t}\delta_t\Big). \tag{205}$$

Step 2: We need to show that the drift satisfies (107), i.e.

$$b(x,t) = \frac{1}{2}\frac{\|U_x^X(x,t)\lambda_t + \sigma_t^{tr}\sigma_t(\sigma_t^{tr}\sigma_t)^{-1}v_x(x,t)\|^2}{U_{xx}^X(x,t)} - \tilde{U}^{(c)}\big(U_x(x,t),t\big). \tag{206}$$

We compute the derivatives for the above formula

$$U_x^X(x,t) = u_x^{(1)}(\frac{x}{N_t},t)\frac{Z_t}{N_t} = \big(\frac{x}{N_t}\big)^{p-1}e^{f(t)}\frac{Z_t}{N_t} \tag{207}$$

$$U_{xx}^X(x,t) = u_{xx}^{(1)}(\frac{x}{N_t},t)\frac{Z_t}{N_t^2} = (p-1)\big(\frac{x}{N_t}\big)^{p-2}e^{f(t)}\frac{Z_t}{N_t^2} \tag{208}$$

$$U_x^C(c,t) = u_x^{(2)}(\frac{c}{N_t},t)\frac{Z_t}{N_t} = \big(\frac{c}{N_t}\big)^{p-1}Ke^{f(t)}\frac{Z_t}{N_t}. \tag{209}$$

We recall that the convex conjugate is given by

$$\tilde{U}^{(c)}(U_x^X(x,t),t) = U^C(I^C(U_x^X(x,t)),t) - U_x^X(x,t)I^C(U_x^X(x,t)). \tag{210}$$

We compute the inverse marginal $I^C(\cdot)$, given by the spatial inverse of $U_x^C(\cdot,\cdot)$, and evaluate it at the spatial derivative of $U^X(\cdot,\cdot)$

$$I^C(U_x^X(x,t)) = (U_x^C)^{-1}\big(U_x^X(x,t)\big) \tag{211}$$

$$= xK^{-\frac{1}{p-1}}, \tag{212}$$

which gives that

$$\tilde{U}^{(c)}(U_x^X(x,t),t) = \frac{(1-p)}{p}K^{1-q}\big(\frac{x}{N_t}\big)^p e^{f(t)}Z_t. \tag{213}$$

We also compute the spatial derivative of the volatility process above (205)

$$v_x(x,t) = \frac{\partial}{\partial x}\Big(Z_t\Big(u^{(1)}\phi_t - u_x^{(1)}\frac{x}{N_t}\delta_t\Big)\Big) \tag{214}$$

$$= Z_t\Big(u_x^{(1)}(\frac{x}{N_t},t)\frac{\phi_t}{N_t} - u_x^{(1)}(\frac{x}{N_t},t)\frac{\delta_t}{N_t} - u_{xx}^{(1)}(\frac{x}{N_t},t)\frac{\delta_t x}{N_t^2}\Big) \tag{215}$$

$$= Z_t\frac{x^{p-1}}{N_t^p}e^{f(t)}\big(\phi_t - p\delta_t\big). \tag{216}$$

Plugging the above derivatives into (206) gives

$$b(x,t) = \frac{1}{2}\frac{\|Z_t\frac{x^{p-1}}{N_t^p}e^{f(t)}\Big(\lambda_t + \eta_t\big(\phi_t - p\delta_t\big)\Big)\|^2}{(p-1)\frac{x^{p-2}}{N_t^p}Z_te^{f(t)}} + \frac{p-1}{p}K^{1-q}\big(\frac{x}{N_t}\big)^p e^{f(t)}Z_t \tag{217}$$

$$= \big(\frac{x}{N_t}\big)^p\frac{Z_te^{f(t)}}{p}\Big(\frac{1}{2}q\|\lambda_t + \eta_t\big(\phi_t - p\delta_t\big)\|^2 + (p-1)K^{1-q}\Big) \tag{218}$$

$$=: \text{r.h.s.}, \tag{219}$$

where $\eta_t := \sigma_t^{tr}\sigma_t(\sigma_t^{tr}\sigma_t)^{-1}$. Recall $\eta_t\phi_t = \phi_t$ and $\eta_t\delta_t = \delta_t$ by assumption.

Step 3: We now want to equate the drift (204) with r.h.s. from above. For (204) we still need the $t-$derivative, which is given by

$$u_t^{(1)}(\frac{x}{N_t},t) = \frac{1}{p}\big(\frac{x}{N_t}\big)^p e^{f(t)}f'(t). \tag{220}$$

This gives us that the drift (204) is of the form

$$
\begin{aligned}
b(x,t) =& Z_t\Big(\frac{1}{p}\big(\frac{x}{N_t}\big)^p e^{f(t)} f'(t) + \big(\frac{x}{N_t}\big)^{p-1} e^{f(t)} \frac{x}{N_t}\big(\delta_t^{tr}\delta_t - \delta_t^{tr}\lambda_t - \delta_t^{tr}\phi_t\big) + \\
& \frac{1}{2}(p-1)\big(\frac{x}{N_t}\big)^{p-2}\frac{x^2}{N_t^2}\delta_t^{tr}\delta_t e^{f(t)}\Big) && (221)\\
=& \big(\frac{x}{N_t}\big)^p \frac{e^{f(t)}Z_t}{p}\Big(f'(t) + p\big(\delta_t^{tr}\delta_t - \delta_t^{tr}\lambda_t - \delta_t^{tr}\phi_t + \frac{1}{2}(p-1)\delta_t^{tr}\delta_t\Big) && (222)\\
=&: \text{l.h.s.} && (223)
\end{aligned}
$$

We equate $\text{ l.h.s.} \stackrel{!}{=} \text{r.h.s. }$ to deduce that $f(\cdot)$ solves

$$
\begin{aligned}
& f'(t) = \frac{1}{2}q\|\lambda_t + \phi_t - p\delta_t\|^2 + (p-1)K^{1-q} + p\delta_t^{tr}\big(\lambda_t + \phi_t - \frac{1}{2}(p+1)\delta_t\big), \qquad t \geq 0, && (224)\\
& f(0) = 0. && (225)
\end{aligned}
$$

Thus, we obtain that

$$
\begin{aligned}
& U^X(x,t) = \frac{1}{p}\big(\frac{x}{N_t}\big)^p e^{f(t)} Z_t && (226)\\
& U^C(c,t) = \frac{1}{p}\big(\frac{c}{N_t}\big)^p K e^{f(t)} Z_t && (227)\\
& \text{for } f(t) = \int_0^t \Big(\frac{1}{2}q\|\lambda_s + \phi_s - p\delta_s\|^2 + (p-1)K^{1-q} + p\delta_s^{tr}\big(\lambda_s + \phi_s - \frac{1}{2}(p+1)\delta_s\big)\Big)ds, \quad t \geq 0.
\end{aligned}
$$

□

Proof of Corollary 5.2.1. We recall Theorem 5.2 which states that the optimal policies for an Itô-type forward utility pair are given by

$$
\begin{aligned}
\pi_t^* &= (\sigma_t\sigma_t^{tr})^{-1}\sigma_t\Big(R^X(X^*,t)\lambda_t - \frac{1}{U_{xx}^X(X^*,t)}v_x(X^*,t)\Big), && t \geq 0, && (228)\\
c_t^* &= I^C\big(U_x^X(X^*,t)\big), && t \geq 0. && (229)
\end{aligned}
$$

We observe in the proof of Lemma 3 above that Equation (212) gives the optimal consumption policy

$$
c_t^* = K^{1-q}, \tag{230}
$$

since we know that $(1-p)(1-q) = 1$ for $\frac{1}{p} + \frac{1}{q} = 1$. For the optimal investment strategy, we see from Equation (216) in the above proof that the derivative of the volatility process is $v_x(x,t) = Z_t\frac{x^{p-1}}{N_t^p}e^{f(t)}\big(\phi_t - p\delta_t\big)$. The derivatives of the wealth performance process $U^X(\cdot,\cdot)$, given by (207), (208), imply that the risk tolerance function takes the form

$$
R^X(x,t) = \frac{x}{(1-p)}, \qquad t \geq 0. \tag{231}
$$

Thus we plug the above results into the formula (228) to obtain that

$$
\begin{aligned}
\pi_t^* &= (\sigma_t\sigma_t^{tr})^{-1}\sigma_t\Big(\frac{X_t^*}{(1-p)}\lambda_t + \frac{Z_t e^{f(t)}N_t^{-p}(X_t^*)^{p-1}}{Z_t e^{f(t)}N_t^{-p}(X_t^*)^{p-2}(1-p)}(\phi_t - p\delta_t)\Big) && (232)\\
&= (\sigma_t\sigma_t^{tr})^{-1}\sigma_t\Big(\frac{X_t^*}{(1-p)}\big(\lambda_t + \phi_t - \delta_t\big) + X_t^*\delta_t\Big), && (233)
\end{aligned}
$$

which proves the claim. □

A.4 Proof of Theorem 6.1

Proof. The main idea of the proof is to use tools from duality theory. More precisely, we first derive the dual value function, and then invoke the key duality theorem to obtain the primal value function. This then allows us to get the equation for the indifference price, which we then solve.

We start with the derivation of the dual value function (125). For the case of exponential utility, we compute that the convex conjugate is given by $\tilde{U}(s) = \frac{s}{\alpha}\big(\log\big(\frac{s}{\alpha}\big) - 1\big)$, hence

$$v^{(n)}(s, Y_t, t) = \operatorname*{ess\,inf}_{Z\in\mathcal{Z}} \mathbb{E}[\frac{s}{\alpha}\frac{Z_T}{Z_t}\big(\log\left(\frac{s}{\alpha}\frac{Z_T}{Z_t}\right) - 1\big) + s\frac{Z_T}{Z_t} nh(Y_T, S_T)|\mathcal{F}_t] \tag{234}$$

$$= \frac{s}{\alpha}\big(\log\left(\frac{s}{\alpha}\right) - 1\big) + \frac{s}{\alpha}\operatorname*{ess\,inf}_{Z\in\mathcal{Z}} \mathbb{E}[\frac{Z_T}{Z_t}\log\left(\frac{Z_T}{Z_t}\right) + n\alpha h(Y_T, S_T)|\mathcal{F}_t] \tag{235}$$

$$= \tilde{U}(s) + \frac{s}{\alpha}\operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{M}}\{\mathcal{H}_t(\mathbb{Q}|\mathbb{P}) + n\alpha\mathbb{E}^{\mathbb{Q}}[h(Y_T, S_T)|\mathcal{F}_t]\}. \tag{236}$$

Define $\xi_t := \operatorname{ess\,inf}_{\mathbb{Q}\in\mathcal{M}}\{\mathcal{H}_t(\mathbb{Q}|\mathbb{P}) + n\alpha\mathbb{E}^{\mathbb{Q}}[h(Y_T, S_T)|\mathcal{F}_t]$. Then the key theorem of duality (see Section 1 in Delbaen et al. [12], or the time-dependent version given in Monoyios [38], Theorem 4.5 for the case $n = -1$) asserts that

$$u^{(n)}(X_t^\pi, Y_t, t) = \operatorname*{ess\,inf}_{s>0}\{v^{(n)}(s, Y_t, t) + X_t^\pi s\}, \qquad 0 \le t \le T. \tag{237}$$

By first order conditions we deduce that $s^* = \alpha e^{-\alpha X_t^\pi - \xi_t}$, so we get that the primal value function, conditioned on $X_t^\pi = x, Y_t = y$, is given by

$$u^{(n)}(x, y, t) = -\exp\Big(-\alpha x - \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{M}}\{\mathcal{H}_t(\mathbb{Q}|\mathbb{P}) + n\alpha\mathbb{E}^{\mathbb{Q}}[h(Y_T, S_T)|X_t^\pi = x, Y_t = y]\Big). \tag{238}$$

This asserts the first part of the theorem, which has already been established by Delbaen et al. [12], Theorem 1, and the time-dependent version by Monoyios [38], Theorem 4.5, for the case $n = -1$.

Recall the definition of the indifference price (126), which requires that we find the value function for $n = 0$. We deduce that

$$u^{(0)}(x, t) = -e^{-\alpha x - \mathcal{H}_t(\mathbb{Q}^E|\mathbb{P})}, \tag{239}$$

The definition of the indifference price (126) together with (238) and (239) gives that

$$\alpha np^{(n)}(t, x, y) = \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{M}}\{n\alpha\mathbb{E}^{\mathbb{Q}}[h(Y_T, S_T)|\mathcal{F}_t] + \mathcal{H}_t(\mathbb{Q}|\mathbb{P}) - \mathcal{H}_t(\mathbb{Q}^E|\mathbb{P})\}. \tag{240}$$

Using $-\operatorname{ess\,inf} -C = \operatorname{ess\,sup} C$ for some random variable C and Proposition 4.7 in Monoyios [38], which asserts that $\mathcal{H}(\mathbb{Q}_1|\mathbb{Q}_2) = \mathcal{H}_t(\mathbb{Q}_1|\mathbb{P}) - \mathcal{H}_t(\mathbb{Q}_2|\mathbb{P})$, for $t \ge 0, \mathbb{Q}_{1/2} \in \mathcal{M}$, we derive that

$$p^{(n)}(x, y, t) = \begin{cases} \operatorname{ess\,inf}_{\mathbb{Q}\in\mathcal{M}}\{\mathbb{E}^{\mathbb{Q}}[h(Y_T, S_T)|\mathcal{F}_t] + \frac{1}{n\alpha}\mathcal{H}_t(\mathbb{Q}|\mathbb{Q}^E)\}, & n \ge 0, \\ \operatorname{ess\,sup}_{\mathbb{Q}\in\mathcal{M}}\{\mathbb{E}^{\mathbb{Q}}[h(Y_T, S_T)\mathcal{F}_t] - \frac{1}{|n|\alpha}\mathcal{H}_t(\mathbb{Q}|\mathbb{Q}^E)\}, & n \le 0. \end{cases} \tag{241}$$

□

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